ON THE BOLTZMANN EQUATION FOR LONG-RANGE INTERACTIONS

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ABSTRACT. We study the Boltzmann equation without Grad's angular cut-off assumption. We introduce a suitable renormalized formulation, which allows the cross-section to be singular in both the angular and the relative velocity variables. Angular singularities occur as soon as one is interested in long-range interactions, while singularities in the relative velocity variable occur in the study of soft potentials, in particular Coulomb interaction. Together with several new estimates, this new formulation enables us to prove existence of weak solutions, and to give a proof of a conjecture by Lions (appearance of strong compactness) under general, fully realistic assumptions.

CONTENTS

1. INTRODUCTION

Since the work of DiPerna and Lions [23] on the Cauchy problem for the Boltzmann equation, ten years ago, it has been a well-known open problem to extend their theory to physically realistic long-range interactions – one major motivation coming from plasma physics, where Coulomb interactions naturally arise. Here we shall give an almost complete solution to this problem, and introduce several new tools on

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this occasion. Before explaining our methods and results, let us give a detailed presentation of the problem and its motivations. The equation that we study here is the Boltzmann equation

$$(1) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f),$$

where the unknown $f(t, x, v)$ is a nonnegative integrable function. For each time $t \geq 0$, $f(t, \cdot, \cdot)$ stands for the density of particles in phase space : position $x \in \mathbb{R}^N$ or $\mathbb{T}^N$ (the N-dimensional torus), and velocity $v \in \mathbb{R}^N$, $N \geq 2$. Moreover, Q is the Boltzmann collision operator, which acts only on the velocity dependence of f (this reflects the physical assumption that collisions are localized in space and time),

$$(2) \qquad Q(f, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma)(f' f'_* - f f_*).$$

Here $f' = f(v')$ and so on (t, x are only parameters), and the formulas

$$(3) \qquad \begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\[2mm] v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma \end{cases}$$

yield one convenient parametrization of the set of solutions to the laws of elastic collision

$$(4) \qquad \begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \end{cases}$$

The velocities v' and v'_* should be thought of as the possible precollisional velocities of particles which in a collision process acquire new velocities v and v_*. On physical grounds, the nonnegative (a.e. finite) weight function $B(v - v_*, \sigma)$, called the cross-section, is assumed to depend only on $|v-v_*|$ (modulus of the relative velocity) and on the scalar product $(\frac{v-v_*}{|v-v_*|}, \sigma)$ (cosine of the deviation angle). For a given interaction model, the cross-section can be computed in a semi-explicit way by the solution of a classical scattering problem, see for instance [13].

We shall write indifferently $a \cdot b$ or (a, b) for the scalar product of a and b, and use the notations

$$(5) \qquad k = \frac{v - v_*}{|v - v_*|}, \qquad k \cdot \sigma = \cos\theta, \qquad 0 \leq \theta \leq \pi.$$

Without loss of generality we shall assume that $B(v-v_*, \sigma)$ is supported in the set $(0 \leq \theta \leq \pi/2)$, i.e. $(v - v_*, \sigma) \geq 0$. If not, we can reduce to

this case upon replacing B by its symmetrized version

$$\overline{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)]1_{(v-v_*,\sigma)>0}.$$

This is a consequence of the undiscernability of particles. More complicated models are possible, for instance with a self-consistent mean-field force, but this does not affect the key analytical difficulties related to equation (1) : see the discussion and arguments in Lions [37].

  In all the sequel, we shall consider only the case when x varies in $\mathbb{R}^N$, and we simply point out that all the analysis adapts trivially to the case of the torus. We also fix once for all an *arbitrary* time interval $[0, T]$.

  Known a priori estimates for solutions of (1) are based mainly on the following fundamental physical laws :

  - *conservation of the total mass, momentum, energy* :

$$\frac{d}{dt} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \begin{pmatrix} 1 \\ v \\ \frac{|v|^2}{2} \end{pmatrix} dx \, dv = 0;$$

- *decrease of entropy* (Boltzmann's celebrated H-theorem), which follows from the well-known formal identity

$$(6) \quad \frac{d}{dt} \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \log f(t, x, v) \, dx \, dv$$

$$= -\frac{1}{4} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^{2N}} dv \, dv_* \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma)(f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*}.$$

Notations : Given a nonnegative function $f(x, v)$, we shall denote by

$$(7) \qquad\qquad H(f) = \int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} f \log f$$

the standard Boltzmann H-functional, or entropy, and we set

$$L \log L(\mathbb{R}^N_x \times \mathbb{R}^N_v) = \left\{ f \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v); \int_{\mathbb{R}^N \times \mathbb{R}^N} |f| \log(1 + |f|) \, dx \, dv < +\infty \right\},$$

with the associated natural Orlicz norm. Also, given a nonnegative function $f(v)$, we let

$$(8) \quad D(f) = \frac{1}{4} \int_{\mathbb{R}^{2N}} dv \, dv_* \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma)(f'f'_* - ff_*) \log \frac{f'f'_*}{ff_*}$$

stand for the (nonnegative) entropy dissipation functional associated to f.

Finally, using the properties of the transport operator, it is also very easy to obtain a (local in time) estimate on

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \, |x|^2 \, dx \, dv.$$

Here is a rough bound, derived from $(d/dt) \int f |x - vt|^2 \, dx \, dv = 0$:

$$\int f(t, x, v)|x|^2 \, dx \, dv \leq 2 \int f(0, x, v)|x|^2 \, dx \, dv + 2t^2 \int f(0, x, v)|v|^2 \, dx \, dv.$$

At the end of the eighties, DiPerna and Lions [22, 23] showed that the resulting estimates, namely

$$(9) \quad f \in L_t^\infty \big([0, T]; L_{x,v}^1 ((1 + |v|^2 + |x|^2) \, dx \, dv) \cap L \log L(\mathbb{R}_x^N \times \mathbb{R}_v^N) \big),$$

$$(10) \qquad\qquad D(f) \in L^1([0, T] \times \mathbb{R}_x^N)$$

were sufficient to build a mathematical theory of weak solutions – there called renormalized solutions – to equation (1), in the following sense.

Definition 1. *A nonnegative function $f \in C(\mathbb{R}^+; L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$ is called a* renormalized solution *of the Boltzmann equation (1) if for all nonlinearity $\beta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, such that $\beta(0) = 0$, $\beta'(f) \leq C/(1 + f)$,*

$$(11) \qquad\qquad \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) = \beta'(f) Q(f, f),$$

in the sense of distributions.

Their main assumption on the cross-section was *Grad's angular cut-off*, namely that *the cross-section be integrable*, locally in all variables. More precisely, they assumed

$$(12) \qquad\qquad A(z) \equiv \int_{S^{N-1}} B(z, \sigma) \, d\sigma \ \in L_{\mathrm{loc}}^1(\mathbb{R}^N),$$

together with a condition of mild growth of A as $|z| \to \infty$: essentially,

$$(13) \qquad\qquad A(z) = o(|z|^2) \qquad \text{as } |z| \to \infty.$$

Apart from weak compactness estimates, the analysis of DiPerna and Lions made crucial use of

- 1) a renormalized formulation, which is a distributionally meaningful definition of $\beta'(f) Q(f, f)$ under the above a priori estimates and Grad's angular cut-off assumption;

- 2) the so-called averaging lemmas [28, 29, 26], which express compactness or smoothness properties in (t, x) of the velocity-averages of solutions of transport equations.

Here is how the renormalized formulation was achieved : following a longstanding tradition in kinetic theory, one splits the collision

operator (2) into a positive ("gain") and a negative ("loss") part :
$Q = Q^+ - Q^-$, where

$$Q^+(f, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma) f' f'_*,$$

$$Q^-(f, f) = f(A *_v f).$$

Then, one notes that under the assumptions on β in Definition 1,

(14)
$$\beta'(f) Q^-(f, f) \le \frac{Cf}{1 + f}(A * f)$$

lies in $L^\infty([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_v))$. Finally, integrating equation (11) in all variables, one finds the additional a priori estimate

(15)
$$\beta'(f) Q^+(f, f) \in L^1_{\mathrm{loc}}([0, T] \times R^N_x \times \mathbb{R}^N_v).$$

(There is also a more complicated proof of (15), based on the use of the entropy dissipation estimate).

This simple manipulation is enough to define us "meaningful" weak solutions of (1). But much work remains to be done in order to prove that these solutions are *stable*, in the following sense. From any sequence $(f^n)_{n \in \mathbb{N}}$ of renormalized solutions satisfying the natural bounds (finite mass, energy, entropy, entropy dissipation)

(16) $\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \int_{\mathbb{R}^N \times \mathbb{R}^N} f^n \left[1 + |v|^2 + |x|^2 + \log f^n(t, x, v) \right] dx \, dv < +\infty,$

(17) $\sup_{n \in \mathbb{N}} \int_0^T \int_{\mathbb{R}^N_x} D(f^n) \, dx \, dt < +\infty,$

one can extract a subsequence converging weakly to a renormalized solution. A simple variant of these arguments shows that, up to extraction, sequences of *approximate* solutions to (1) converge to renormalized solutions to (1), and of course this is enough to prove existence of such solutions.

Yet, the DiPerna-Lions theorem is certainly best thought of as a — quite unexpected — *stability* result under weak convergence. Nothing is known as concerns the question of propagation of smoothness for these solutions. The only partial result in this direction is due to Lions [35, 36], who proved that a sequence of renormalized solutions (f^n) is strongly (relatively) compact in $L^1([0, T] \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ if and only if the sequence of corresponding initial data (f_0^n) is strongly (relatively) compact in $L^1(\mathbb{R}^N_x \times \mathbb{R}^N_v)$. In other words, **in the cut-off case, no oscillations develop unless they are present from the beginning**. We summarize these results with the following Theorem, due to DiPerna and Lions :

Theorem 1. *Assume Grad's angular cut-off* (12)–(13). *Let* $(f^n)_{n \in \mathbb{N}}$ *be a sequence of renormalized solutions of the Boltzmann equation with respective initial datum* (f_0^n), *satisfying uniform estimates of mass, energy and entropy, in the sense of* (16). *Assume without loss of generality that* $f^n \longrightarrow f$ *weakly in* $L^p([0, T], L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$ (1 ≤ p < +∞). *Then*

(i) f *is a renormalized solution of the Boltzmann equation;*

(ii) *let* f_0 *denote* $f(0, \cdot, \cdot)$; *then*

$f^n \to f$ *   strongly   if and only if* $f_0^n \to f_0$ *strongly in* $L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)$.

DiPerna and Lions [25] also proved that if the solutions f^n satisfy the entropy dissipation estimate

$$(18) \qquad \int_0^T dt \int_{\mathbb{R}_x^N} dx \, D\big(f^n(t, x, \cdot)\big) \leq H(f_0^n) - H(f^n(T, \cdot, \cdot)),$$

then the limit solution f also satisfies this estimate.

Even though, up to this date, many open questions remain on the subject of renormalized solutions (uniqueness, smoothness, energy conservation), the DiPerna-Lions result has enabled the study of several physical phenomena, in particular the hydrodynamical limit whose analysis was later performed by Bardos, Golse and Levermore [9], with recent spectacular developments by Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond [10, 27, 30, 40]. Also the result of propagation of compactness is interesting from the physical point of view.

Yet the main restriction on the cross-section, namely Grad's angular cut-off (12), is definitely not satisfactory from the physical point of view — though extremely common in the field ! Indeed, as soon as one considers *long-range* interactions, even with a very fast decay at infinity, this assumption is not satisfied, and the integral in the left-hand side of (12) is infinite (except maybe for $z = 0$). The typical example is that of inverse s-power repulsive forces in dimension 3, which give rise to a cross-section

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(k \cdot \sigma),$$

with $\gamma = (s - 5)/(s - 1)$ and

$$\sin \theta \, b(\cos \theta) \sim K\theta^{-1-\nu} \text{ as } \theta \to 0, \qquad \nu = \frac{2}{s - 1} > 0, \quad K > 0$$

(see for instance [14]). Here the factor $\sin \theta$ corresponds to the Jacobian factor for the integration in spherical coordinates. Thus the function $\sin \theta \, b(\cos \theta)$ presents a *nonintegrable* singularity as $\theta \to 0$: this regime corresponds to *grazing collisions*, i.e. collisions in which particles are

hardly deviated. Physically speaking, these are the collisions between particles that are microscopically very far apart, with a large impact parameter.

Another complication (often worse, as shown by the experience from the spatially homogeneous theory [45], and as we shall experience here again) arises when trying to deal with the Coulomb potential : for $s = 2$ in dimension $N = 3$ as above, one finds a cross-section behaving like $|v - v_*|^{-3}$ in the relative velocity variable, hence *not locally integrable* as a function of the relative velocity (this is the limit, borderline case). The DiPerna-Lions formulation cannot handle this case which is one of the most important from the physical points of view.

It is our purpose here to treat both singularities, and extend the DiPerna-Lions theory to very general, physically realistic long-range interactions, including the Coulomb potential as a limit case, in a sense which will be made precise later on. This extension will be made possible by three new tools :

- a sharp understanding of the smoothness estimates associated with the entropy dissipation in presence of grazing collisions;

- a better understanding of the cancellation effects associated with the symmetries of the Boltzmann kernel;

- a new and much subtler procedure of renormalization.

In particular, we shall show how certain simple objects govern several properties of the Boltzmann collision operator. The most important is the *cross-section for momentum transfer*, M, which is defined by

$$(19) \qquad \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma)[v - v'] = 2(v - v_*)M(|v - v_*|).$$

Finiteness a.e. of M is a necessary condition for the Boltzmann collision operator to make sense [46]. Here we shall show that it is also essentially sufficient, under some weak additional assumptions. From the physical point of view, this is very good news, because the cross-section for momentum transfer is one of the basic quantities in the theory of binary collisions (and its computation via experimental measurements is a well-developed topic).

On the other hand, the price to pay for such an extension will be a weakening of the notion of renormalized solution, as follows.

Definition 2. *We shall say that a nonnegative function*

$$f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v)) \cap L^\infty(\mathbb{R}^+; L^1((1 + |v|^2 + |x|^2) \, , dx \, , dv))$$

is a renormalized solution of the Boltzmann equation with a defect measure, *if for all nonlinearity* $\beta \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ *satisfying*

(20) $\qquad \beta(0) = 0, \qquad 0 < \beta'(f) \leq \dfrac{C}{1+f}, \qquad \beta''(f) < 0,$

the inequality

(21) $\qquad \dfrac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) \geq \beta'(f) Q(f, f)$

holds in the sense of distributions, together with the mass-conservation condition

(22) $\quad \forall t \geq 0, \qquad \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \, dx \, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(0, x, v) \, dx \, dv.$

Example : A typical choice for $\beta(f)$ will be $f/(1 + \delta f)$, $\delta > 0$.

At this level we do not yet make precise the sense of $\beta'(f) Q(f, f)$ in (21), and we only wish to point out that equation (21), combined with the mass-conservation condition (22), really defines a weak solution and not an upper solution. This can be seen by the following crude formal argument, which appears first in the work of DiPerna and Lions [22] on the Fokker-Planck-Boltzmann equation : let μ be the nonnegative "defect" measure equal to the difference of the left-hand side and the right-hand side of (21). If f were smooth, multiplication of both sides of (21) by $1/\beta'(f)$ and integration in all variables would yield

$$\int f(T, x, v) \, dx \, dv = \int f(0, x, v) \, dx \, dv + \int_{[0,T] \times \mathbb{R}^N \times \mathbb{R}^N} d\mu(t, x, v),$$

and the mass-conservation would imply the vanishing of μ.

Definition 2 may seem unsatisfactory, in that it introduces an unknown object (the defect measure), with a priori no physical meaning at all. Thus it would be desirable to prove, if possible, that the defect measure automatically vanishes, as is the case for the simpler Fokker-Planck-Boltzmann model studied by DiPerna and Lions in [22].

However, we must warn the reader that such an achievement would only be for the sake of elegance : no relevant physical conclusion, no new estimate would be provided by this result – which seems far out of reach, because of technical problems that we describe in the Appendix.

On the other hand, our definition is strong enough to prove such statements as the strong compactification effect, or the Landau approximation. We also expect that some of the recent progress about the fluid dynamics limits for the Boltzmann equation (see the above-mentioned references) can be treated in the framework of renormalized

solutions with a defect measure (in particular the methods of [10], which already involve a defect measure).

Thus, we shall build here a theory of renormalized solutions with a defect measure, and again, following Lions [35], we wish to emphasize that, even if such a concept of solutions may seem quite weak, the results that we shall obtain make sense, and are nontrivial, even for hypothetical strong solutions. In particular, we shall be able to prove two fundamental qualitative properties of the solutions to the Boltzmann equation, under very general assumptions (spatially inhomogeneous setting; no restriction on the size of the data).

1) the first one, which is the content of the present paper, is the fact that **singular cross-sections induce an immediate damping of oscillations** (Lions' conjecture), which is at the opposite of the propagation result in the cut-off case. A precise statement can be found in the next section.

2) Our second main result, discussed in our companion paper [7], is the first rigorous mathematical justification of the Landau approximation in plasma physics. Roughly speaking, this means the replacement of Boltzmann's collision operator for the so-called screened Debye potential, by the Landau collision operator for Coulomb potential, in the limit when the Debye length is very large. Such a result was proven in Villani [45] in the homogeneous situation, i.e. when distribution functions do not depend on the position x, and we shall extend it in [7] to the much more delicate x-dependent problem. Since the Boltzmann operator is meaningless for "bare" Coulomb interaction, this limit procedure is the right framework to deal with Coulomb collisions.

We end this introduction by briefly discussing recent progress in the study of longe-range interactions and singular cross-sections.

Until the last few years, and despite substantial efforts, extremely little was known concerning the Boltzmann equation without cut-off. Among the main exceptions were the remarks by Pao [41] and Klaus [33] on the linear Boltzmann operator without cut-off. Some of these remarks were actually misleading : for instance Klaus asserts the impossibility of using the Boltzmann equation for inverse s-power forces with $s \leq 3$. The other important contribution to the subject was the proof by Arkeryd [8] of existence of weak solutions to the spatially homogeneous Boltzmann equation without cut-off, also in the case $s > 3$. Then, starting from the mid-nineties, a great deal of works appeared on the subject, and a lot of progress was made in several parallel directions :

1. The Cauchy problem associated with the Landau equation was studied by Lions [38], who proved strong compactness properties of sequences of approximate solutions, and by Villani [44], who deduced from this result the existence of suitable weak solutions. The study of Lions led him to his conjecture on the appearance of strong compactness for the Boltzmann equation without cut-off [36], which he further explored in his more recent note [39]. Here we shall re-use his main ideas, and, thanks to the new tools mentioned before, prove his conjecture in full generality.

2. The theory of weak solutions in the spatially homogeneous case, initiated by Arkeryd, was extended independently by Goudon [31] (for $s \geq 7/3$ in dimension 3) and by Villani [45] (for $s > 2$). In the last work, the introduction of so-called H-solutions, with a formulation based on the entropy dissipation, allowed us to treat Coulomb interaction (in fact, angular singularities together with singularities in the relative velocity like $|v - v_*|^\gamma$, $\gamma > -4$ independently of the dimension), and justify the Landau approximation.

3. The study of qualitative properties of solutions to the spatially homogeneous Boltzmann equation without cut-off was first addressed in the works of Desvillettes [17, 18, 19] and his student Proutière [42]. Graham and Méléard [32] managed to recover the results of Desvillettes for the one-dimensional Kac model by a purely probabilistic method relying on the Malliavin calculus. In all these works it is proven that in some particular regimes, the Boltzmann equation without cut-off has smoothing properties, which the Boltzmann equation with cut-off does not enjoy. Further research on this topic is being performed by Desvillettes and Wennberg, and by the second author as well.

4. The functional counterparts of these smoothing properties at the level of the entropy dissipation (8) were studied by Lions [39], Villani [43], Alexandre [3, 4]. These papers show that the entropy dissipation associated with a singular kernel controls the smoothness in the velocity variable. They all consider different assumptions and obtain different conclusions. Finally, optimal and extremely general results were obtained in the recent study [6], which is a joint work with Desvillettes and Wennberg, and actually a preliminary to the present paper. It is a nice feature of the proofs therein that they invoke Bobylev's idea of using the Fourier transform for the study of the Boltzmann equation.

5. The mere structure of the collision operator was studied in Alexandre [1, 2]. There it was shown how to view the Boltzmann collision operator in a pseudo-differential formalism, at least if the cross-section

is smooth in the relative velocity variable. One high point of this program was the definition of the first renormalized formulation of the Boltzmann equation without cut-off, based on the theory of pseudo-differential operators [5, 4]. The present work extends considerably the domain of application of the renormalization, and avoids any use of pseudo-differential operators, allowing fully general cross-sections.

On the whole, the picture of grazing collisions now appears much clearer than it ever was before. A general "moral" that one should glean from all these works, as already stated in [6], is the following formal statement. *For a given nonnegative function $g \in L^1$, the linear operator*

$$(23) \qquad f \longmapsto Q(g, f) = \int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\sigma \, B(v - v_*, \sigma)(g'_* f' - g_* f),$$

with $B(v - v_*, \sigma) = \Phi(|v - v_*|) b(k \cdot \sigma)$, $\sin^{N-2} \theta \, b(\cos \theta) \sim K\theta^{-1-\nu}$, behaves (from the smoothness point of view) like the fractional diffusion operator $-(-\Delta)^{\nu/2}$.

In the special case of three-dimensional Maxwellian molecules (inverse 5-power forces), and in a linear context, this heuristic rule goes back to Cercignani [13] thirty years ago. From the physical point of view, it means that *"real" interaction processes are neither purely collisional nor purely diffusive, but somewhat in between.*

By the way, in this paper as well as in our previous work [6], it will be important to study properties not only of the quadratic Boltzmann operator (2), but also of the *bilinear* Boltzmann operator $Q(g, f)$ defined by (23). Two particular cases will be of special interest : the linear operator $f \mapsto Q(f, 1)$, and the adjoint of the linear operator $f \mapsto Q(\delta_{v_*}, f)$.

We also wish to remark that apart from the angular singularity, many open questions remain as long as one is interested in strong singularities of the cross-section in the relative velocity variable ("kinetic" singularities). Let us only mention that even in the spatially homogeneous case, it is not known whether solutions of the Landau equation for Coulomb potential (which is a universally accepted model for collisions in a plasma) may develop singularities in finite time. And in this work, even though we are able to cover basically all interesting non-cutoff potentials, we paradoxically fail in the case of cross-sections which present a nonintegrable kinetic singularity but no angular singularity. To our knowledge, such cases do not occur in physically realistic models, but are sometimes introduced by physicists as approximate models...

The paper is organized as follows. In the next section, we detail our assumptions on the cross-section $B(v - v_*, \sigma)$, and then state our main results. In section 3 we present our renormalized formulation, and consider with particular attention the borderline case of a singularity in $|v - v_*|^{-N}$. In section 4, we recapitulate known a priori estimates, show that strong compactness appears for positive times, and prove a stability theorem which implies our main results. Finally, in section 5, we discuss a new conjecture about the possible regularizing effects of a borderline kinetic singularity.

2. Assumptions on the cross-section and main results

Throughout the paper, we shall sometimes abuse notations by writing

$$B(v - v_*, \sigma) = B(|v - v_*|, \cos \theta),$$

where $\cos \theta = (\frac{v - v_*}{|v - v_*|}, \sigma)$, and $0 \leq \theta \leq \pi/2$. Recall that we got rid of collisions with deflexion angle larger than $\pi/2$ by a simple symmetrization trick. It should be helpful to keep in mind the picture of collisions, as recalled in fig. 2

It is sometimes mentioned in the physical literature that the most meaningful quantity to associate with B is the so-called cross-section for momentum transfer, that we shall define as

$$(24) \quad M(|v - v_*|) \equiv \int_{S^{N-1}} B(v - v_*, \sigma)(1 - k \cdot \sigma) \, d\sigma$$

$$= |S^{N-2}| \int_0^{\frac{\pi}{2}} B(|v - v_*|, \cos \theta)(1 - \cos \theta) \sin^{N-2} \theta \, d\theta.$$

Note first that since $1 - \cos \theta = 2 \sin^2(\theta/2)$ vanishes up to order 2 for θ close to 0, this quantity may be (and in fact, is typically) finite even in the non cut-off case.

As we mentioned in the introduction,

$$\int_{S^{N-1}} B(v - v_*, \sigma)(v - v') \, d\sigma = \frac{1}{2}(v - v_*) M(|v - v_*|),$$

so that M really measures the mean quantity of momentum transferred via collisions.
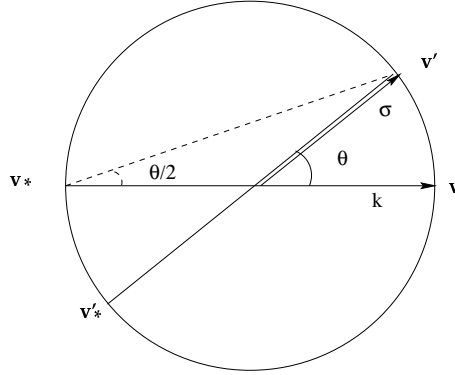
FIGURE 1.   The geometry of collisions

Remarkably, as first emphasized in Villani [45, 46], suitable conditions on M are *exactly* what we need to give meaning to the Boltzmann operator (as an operator acting on functions of the variable v), and to develop a theory of weak solutions in the spatially homogeneous case. On the other hand, if, for instance, $B(v - v_*, \sigma) = |v - v_*|^\gamma \, b(k \cdot \sigma)$ with $\int b(k \cdot \sigma)(1 - k \cdot \sigma) \, d\sigma = +\infty$, then the Boltzmann operator plainly does not make distributional sense, see [46, Part I, Appendix A].

By analogy with the cut-off theory, it would be natural to require that M define a locally integrable function. Here, in a spatially inhomogeneous context, we shall need an extra condition which ensures some very mild regularity of B in the relative velocity variable. Let us define, for $z \neq 0$,

$$(25) \qquad B'(z, \sigma) = \sup_{1 < \lambda \leq \sqrt{2}} \frac{|B(\lambda z, \sigma) - B(z, \sigma)|}{(\lambda - 1)|z|},$$

and, in the same way as (24),

$$(26) \qquad M'(|v - v_*|) = \int_{S^{N-1}} B'(v - v_*, \sigma)(1 - k \cdot \sigma) \, d\sigma.$$

The size of the nonnegative function M' measures in a very mild sense the regularity of B in the relative velocity variable.

We shall show that a simple sufficient condition for a mathematical treatment of the Boltzmann equation is that both $M(|z|)$ and $|z|M'(|z|)$ be locally integrable. That this condition is extremely weak can be seen from the fact that it allows not only smooth (say, Lipschitz) functions,

but also singular power laws $|v - v_*|^\gamma$ with $\gamma > -N$, since

$$|z| \sup_{1 < \lambda \le \sqrt{2}} \frac{|\lambda z|^\gamma - |z|^\gamma}{(\lambda - 1)|z|} = |z|^\gamma \sup_{1 < \lambda \le \sqrt{2}} \frac{\lambda^\gamma - 1}{\lambda - 1} = C_\gamma |z|^\gamma$$

is locally integrable if $\gamma > -N$.

The assumption that M, $|z|M'$ be locally integrable is quite general. Yet, by requiring it, we would overlook the most interesting case, namely cross-sections presenting a *borderline singularity* like $|v - v_*|^{-N}$. In three dimensions, this is the exponent for Coulomb potential. We shall be able to include this limit case thanks to a striking cancellation property. Thus we can relax the assumptions on M and M' to take into account only that part of B which is, loosely speaking, "not borderline". We emphasize that this extension is nontrivial even in the cut-off case. The fact that borderline singularities may be allowed in a cut-off context was first suggested by the formal study in Alexandre [1], based on pseudo-differential operators. This leads us to our main assumption on B.

Assumption I. *(At most borderline singularity) Assume that*

$$(27) \qquad B(z, \sigma) = \frac{\beta_0(k \cdot \sigma)}{|z|^N} + B_1(z, \sigma), \qquad k = \frac{z}{|z|},$$

for some nonnegative measurable functions β_0 and B_1, and define

$$(28) \qquad \mu_0 = \int_{S^{N-1}} \beta_0(k \cdot \sigma)(1 - k \cdot \sigma) \, d\sigma,$$

$$(29) \qquad M_1(|z|) = \int_{S^{N-1}} B_1(z, \sigma)(1 - k \cdot \sigma) \, d\sigma,$$

$$(30) \qquad M_1'(|z|) = \int_{S^{N-1}} B_1'(z, \sigma)(1 - k \cdot \sigma) \, d\sigma,$$

where

$$B_1'(z, \sigma) = \sup_{1 < \lambda \le \sqrt{2}} \frac{|B_1(\lambda z, \sigma) - B_1(z, \sigma)|}{(\lambda - 1)|z|}.$$

We require that

$$\mu_0 < +\infty, \quad \text{and} \quad M_1(|z|), \ |z|M_1'(|z|) \in L^1_{loc}(\mathbb{R}^N).$$

Remarks :

(1) Assumption I is in fact a simple condition to ensure a more general criterion that will be given in section 3.

(2) As a consequence,

$$M(|z|) = M_1(|z|) + \frac{\mu_0}{|z|^N},$$

$$M'(|z|) \leq M_1'(|z|) + \frac{\mu_0}{|z|^N} \left(\frac{2^{N/2} - 1}{\sqrt{2} - 1} \right).$$

Note in particular that $|z|M(|z|)$ is always integrable; this will be used in the sequel.

Next, we have to take care of the behavior at infinity, in the case of cross-sections that may grow to infinity.

Assumption II. *(Behavior at infinity) For $0 \leq \alpha \leq 2$, let*

$$(31) \qquad M^\alpha(|z|) = \int_{S^{N-1}} B(z, \sigma) \, (1 - k \cdot \sigma)^{\frac{\alpha}{2}} \, d\sigma, \qquad k = \frac{z}{|z|}.$$

We require that for some $\alpha \in [0, 2]$, as $|z| \to \infty$,

$$(32) \qquad M^\alpha(|z|) = o(|z|^{2-\alpha}), \qquad |z|M'(|z|) = o(|z|^2).$$

Remarks :

(1) Obviously, controlling the behavior at infinity of $M^\alpha(|z|)$, $|z|M'(|z|)$ is the same than controlling the behavior of $M_1^\alpha(|z|)$, $|z|M_1'(|z|)$, where M_1^α is defined on the same pattern than (29).

(2) Note that M^2 coincides with M, while M^0 is the usual total cross section (as appearing in the DiPerna-Lions theory). Thus, the first part of condition (32) is an extension of the DiPerna-Lions mild growth condition (13), which corresponds to the case $\alpha = 0$.

(3) We do not require that $M^\alpha(|z|)$ is a.e. finite : our assumption is only for large $|z|$. Moreover, one can allow the slightly more general condition

$$\forall R > 0, \quad \lim_{|z_0| \to \infty} \frac{1}{|z_0|^{2-\alpha}} \int_{|z-z_0| \leq R} M^\alpha(|z|) \, dz = 0,$$

and a similar condition for M'.

(4) In the model case

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta), \; with \; \sin^{N-2}\theta \, b(\cos\theta) \sim K\theta^{-1-\nu},$$

$\nu > 0$, $K > 0$, Assumption II allows $\gamma + \nu < 2$. It is worth noting that this assumption is always fulfilled in the physical

cases of inverse s-power forces in dimension $N = 3$, since

$$\gamma + \nu = \frac{s-3}{s-1} < 1.$$

So far, we have defined only conditions ensuring some control of $B(z, \sigma)$ in the relative velocity variable z, but we still have to define conditions ensuring that it is actually singular in the angular variable.

Assumption III. *(Angular singularity condition)*

(33) $B(z, \sigma) \geq \Phi_0(|z|) b_0(k \cdot \sigma), \qquad k = \dfrac{z}{|z|},$

where Φ_0 is a continuous function, $\Phi_0(|z|) > 0$ if $|z| \neq 0$, and

(34) $\displaystyle\int_{S^{N-1}} b_0(k \cdot \sigma) \, d\sigma = +\infty.$

We wish to emphasize the extreme generality of this assumption. Together with Desvillettes and Wennberg, we have shown in [6] that such a condition is sufficient to entail a "smoothness" estimate in the velocity variable, which is the important point for us. The arguments in [6] rely on the entropy dissipation : see section 4 for a discussion and precise statements.

On the whole, in the model case

$$B(v - v_*, \sigma) = |v - v_*|^\gamma b(\cos\theta), \qquad \sin^{N-2}\theta \, b(\cos\theta) \sim K\theta^{-1-\nu}$$

$(\nu > 0, K > 0)$, our Assumptions I, II and III together allow the following range of parameters :

(35) $\gamma \geq -N, \qquad 0 \leq \nu < 2, \qquad \gamma + \nu < 2.$

Thus, in comparison with the DiPerna-Lions theory, the changes lie in the parameter ν of course, but also in the possibility of letting $\gamma = -N$.

Our first main result is the following.

Theorem 2 (Stability and appearance of strong compactness). *Make assumptions I, II, III. Let (f^n) be a sequence of solutions to the Boltzmann equation, in the sense of Definition 2, satisfying the natural a priori estimates (16), (18). Without loss of generality, assume that $f^n \longrightarrow f$ weakly in $L^p([0, T], L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$ $(1 \leq p < +\infty)$. Then*

(i) f is a solution to the Boltzmann equation, in the sense of Definition 2;

(ii) moreover, automatically, $f^n \longrightarrow f$ strongly.

Compare this statement to Theorem 1. As a consequence of Theorem 2 (in fact, to be more rigorous, of a simple variant), we obtain the following existence result.

Corollary 2.1 (Existence of weak solutions). *Make assumptions I, II, III. Let f_0 be an initial datum satisfying the natural assumption*

$$\int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} f_0(x, v)\big[1 + |v|^2 + |x|^2 + \log f_0(x, v)\big]\, dx\, dv < +\infty.$$

Then, there exists a solution f to the Boltzmann equation, in the sense of Definition 2, with initial datum $f(0, \cdot, \cdot) = f_0$. This solution satisfies, for all $t \geq 0$,

$$(36) \qquad \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v)\, dx\, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f_0(x, v)\, dx\, dv,$$

$$(37) \qquad \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) v\, dx\, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f_0(x, v) v\, dx\, dv,$$

$$(38) \qquad \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) \frac{|v|^2}{2}\, dx\, dv \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} f_0(x, v) \frac{|v|^2}{2}\, dx\, dv,$$

and the entropy inequality

$$(39) \qquad H(f(T, \cdot, \cdot)) + \int_0^T dt \int_{\mathbb{R}^N} dx\, D(f(t, x, \cdot)) \leq H(f_0).$$

Remarks :

(1) Of course, in order for these statements to make sense, we still have to make precise the definition of $\beta'(f)Q(f, f)$, i.e. our renormalized formulation. This will be done in the next section.

(2) Our renormalized formulation also makes sense for a cutoffed interaction, and if we assume that the total cross-section

$$A(|z|) = \int_{S^{N-1}} b(k \cdot \sigma)\, d\sigma$$

satisfies $A \in L^1_{\text{loc}}(\mathbb{R}^N)$, $A(|z|) = o(|z|^2)$ at infinity, then the existence of renormalized solutions, with the formulation presented in section 3 (but with zero defect measure) is a consequence of the results of DiPerna and Lions.

(3) These results do cover the case in which there are both an angular singularity and a borderline singularity in velocity, but they *do not* cover the case in which only the latter is present. See section 5 for a discussion.

3. Renormalized formulation

In this section, we consider a fixed function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the assumptions of Definition 2. In particular, β is 1-to-1, grows at most logarithmically, and β^{-1} is convex. Typical choices are

$$\beta(f) = \frac{1}{\delta} \log(1 + \delta f), \quad \text{or} \quad \beta(f) = \frac{f}{1 + \delta f}, \qquad \delta > 0.$$

In fact, for simplicity, we shall also assume in the sequel that β is bounded, and leave the general case to the reader. The following procedure was inspired by Alexandre [5]. For all nonnegative numbers f, f', let

(40) $\Gamma(f, f') = \beta(f') - \beta(f) - \beta'(f)(f' - f).$

Note that $\Gamma(f, f')$ is nonpositve since β is concave. In fact,

$$\Gamma(f, f') = -\beta'(f)\big[\beta(f')-\beta(f)\big]^2 \int_0^1 ds\,(1-s)(\beta^{-1})''\Big(\beta(f)+s[\beta(f')-\beta(f)]\Big).$$

Example : For $\beta(f) = f/(1 + \delta f)$, one finds

$$\Gamma(f, f') = -\frac{\delta(f' - f)^2}{(1 + \delta f)^2(1 + \delta f')}.$$

Using identity (40), we obtain

$$\beta'(f)(f'f'_* - ff_*) = \beta'(f)f'_*(f' - f) + \beta'(f)f(f'_* - f_*)$$

$$= f'_*[\beta(f') - \beta(f)] - f'_*\Gamma(f, f') + \beta'(f)f(f'_* - f_*)$$

$$= f'_*\beta(f') - f_*\beta(f)$$
$$+ [f\beta'(f) - \beta(f)](f'_* - f_*)$$
$$- f'_*\Gamma(f, f').$$

Now, let $f = f(t, x, v)$ be a density distribution. Replacing f' by $f(t, x, v')$, f'_* by $f(t, x, v'_*)$ and so on, one can write, for almost all (t, x, v),

(41) $\beta'(f)Q(f, f) = (\mathcal{R}_1) + (\mathcal{R}_2) + (\mathcal{R}_3),$

where

(42) $(\mathcal{R}_1) = \big[f\beta'(f) - \beta(f)\big] \int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\sigma\, B(f'_* - f_*),$

(43) $(\mathcal{R}_2) = \int_{\mathbb{R}^N \times S^{N-1}} dv_*\, d\sigma\, B\big[f'_*\beta(f') - f_*\beta(f)\big],$

(44)
$$(\mathcal{R}_3) = -\int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B f'_* \Gamma(f, f').$$

(We have omitted the arguments of B for notational convenience.)

Notation :   We define $\mathcal{S}$ as the linear operator

$$\mathcal{S}f \equiv \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(f'_* - f_*).$$

Thus,

$$(\mathcal{R}_1) = \left[f\beta'(f) - \beta(f) \right] \mathcal{S}f.$$

Remarks :

(1) In our forthcoming work [7], we shall see how well this renormalized formulation is related to the renormalization of the Landau equation, which was introduced by Lions [38] several years ago.

(2) The structure of the renormalized formulation is much clearer when expressed in terms of the (asymmetric) bilinear Boltzmann operator,

$$Q(g, f) = \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(g'_* f' - g_* f).$$

Indeed, note that

$$(\mathcal{R}_2) = Q(f, \beta(f)).$$

Moreover, the same manipulation as above shows that

$$\beta'(f)Q(g, f) = [f\beta'(f) - \beta(f)]\mathcal{S}g + Q(g, \beta(f))$$
$$- \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B \, g'_* \Gamma(f, f').$$

This formula should actually be taken as the definition of the renormalization of the bilinear Boltzmann operator without cut-off.

In the next two subsections, we shall show successively that $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$ make sense in $\mathcal{D}'([0, T] \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$ for any distribution function f satisfying the a priori bound

(45)
$$\sup_{t \in [0,T]} \int f(t, x, v)(1 + |v|^2) \, dx \, dv < +\infty.$$

Then, by integration of the equation in renormalized form, this will lead to a bound on $(\mathcal{R}_3)$, and we shall be able to complete our renormalized formulation.

3.1. **Symmetry-induced cancellation effects.**

We consider the term $(\mathcal{R}_1)$. Despite the nonintegrable singularity in B, it would be easy to give a sense to $\int dv_* \, d\sigma \, B(f'_* - f_*)$ if the function f was smooth, and the singularity in B not too strong. Indeed, grazing collisions occur for $v'_* \simeq v_*$, hence $f'_* - f_*$ should vanish to order 1. In fact, due to symmetries, *there is a cancellation of order 2 in the integral, even if f is not smooth.* This is the content of the following proposition.

Proposition 3 (Cancellation Lemma). *Let B be a nonnegative measurable kernel. Then, for a.a. $v \in \mathbb{R}^N$,*

$$(46) \qquad \mathcal{S}f \equiv \int_{\mathbb{R}^N \times S^{N-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(f'_* - f_*) = f *_v S,$$

where
(47)
$$S(|z|) = |S^{N-2}| \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \left[\frac{1}{\cos^N(\theta/2)} B\left(\frac{|z|}{\cos(\theta/2)}, \cos\theta \right) - B(|z|, \cos\theta) \right].$$

In particular, if B satisfies Assumption I, then

$$(48) \qquad\qquad\qquad S(|z|) = \lambda \delta_0 + S_1(|z|),$$

where δ_0 is the Dirac mass at the origin,

$$\lambda = -|S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \beta_0(\cos\theta) \log \cos(\theta/2) \, \sin^{N-2}\theta \, d\theta,$$

and S_1 is a locally integrable function,

$$|S_1(|z|)| \leq \frac{2^{\frac{N-4}{2}}}{\cos^2(\pi/8)} \big[N M_1(|z|) + |z| M'_1(|z|) \big].$$

Remarks :
  (1) Of course, $\mathcal{S}f = Q(f, 1)$.
  (2) If B is homogeneous of degree $-N$ in the relative velocity variable, then at first sight the kernel S is identically 0. But this hides the fact that B is not integrable, and that $\mathcal{S}$ has to be defined as a *principal value operator*,

$$\mathcal{S}f = \lim_{\varepsilon \to 0} \int dv_* \, d\sigma \, B(v - v_*, \sigma) 1_{|v - v_*| \geq \varepsilon} \, 1_{\theta \geq \varepsilon} \, (f'_* - f_*).$$

Thus the convolution kernel S will in general be a Radon measure. In fact, in such situations, the result of Proposition 3 should be considered as the *definition* of the left-hand side. At the level of the Boltzmann equation, this definition is justified by the fact that all expressions given above coincide as long as f is smooth (in which case $Q(f, f)$ is easily given a sense via Taylor expansions of $f' f'_* - f f_*$ for $v' \simeq v$, $v'_* \simeq v_*$).

(3) Using the inequality $-\log u \leq (1/u) - 1$, we find

$$0 \leq \lambda \leq \frac{|S^{N-1}|}{4 \cos^2(\pi/8)} \, \mu_0.$$

(4) In the case where the kinetic cross-section is a power law, $B(z, \sigma) = |z|^\gamma b(\cos\theta)$, $\gamma \geq -N$, then the integrand in (47) is always non-negative.

(5) In fact, instead of Assumption I, we could impose that S defined by (47) is a locally bounded measure. This kind of assumption will be useful to discuss the Landau approximation.

(6) Similar lemmas appear in previous works by both authors, and also implicitly in former work by Desvillettes [19]. This result is stated here for the first time at such a level of generality. The proof that we present follows quite closely the argument in Villani [43], and part of it was reproduced in the joint work [6], in order to make it self-contained.

Before proving Proposition 3, we state the following corollary.

Corollary 3.1. *Let B satisfy assumptions I and II, and let f satisfy* (45). *Then, $(\mathcal{R}_1)$ defined by* (42) *lies in $L^\infty([0, T]; L^1(\mathbb{R}_x^N \times B_R(v)))$, for all $R > 0$, where $B_R(v) = \{v \in \mathbb{R}^N, \ |v| \leq R\}$.*

Proof of Corollary 3.1. Since $f\beta'(f) \in L^\infty$ by assumption, we only need to show that $f * S \in L^1(\mathbb{R}_x^N \times B_R(v))$. But

$$\|f * S\|_{L^1(\mathbb{R}_x^N \times B_R(v))} \leq \int_{|v| \leq R} f(x, v_*)|S(v - v_*)| \, dv \, dv_* \, dx$$

$$\leq \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} dv_* \, f(x, v_*) \int_{|z + v_*| \leq R} dz \, |S(z)|$$

$$\leq \|f\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)} \|S\|_{TV(|z| \leq R + R')}$$
$$+ \frac{1}{(R')^2} \left[\int dx \, dv_* f(x, v_*)|v_*|^2 \right] \sup_{|v_*| \geq R'} \int_{|z + v_*| \leq R} dz \, |S(z)|,$$

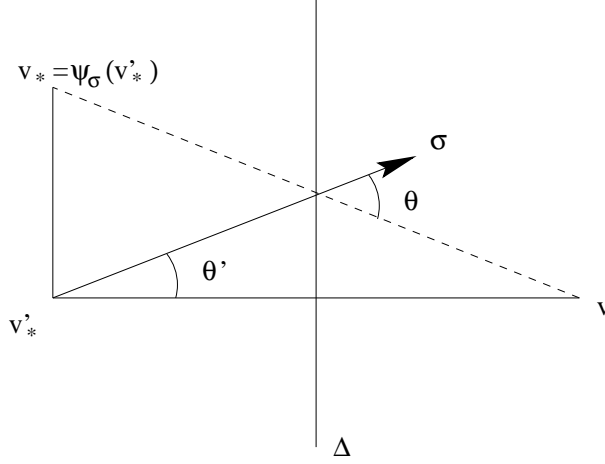FIGURE 2. Δ is the mediatrix of (v, v'_*); $\cos\theta = k \cdot \sigma$; $\cos\theta' = k' \cdot \sigma = \cos(\theta/2)$.

for all $R' > 0$. Here $\|\cdot\|_{TV}$ denotes the norm in total variation. From the Cancellation Lemma and Assumption II, we have $|S(z)| = o(|z|^2)$ as $|z| \to \infty$, and the above expression is finite for R' large enough. $\square$

Proof of Proposition 3. Since $\mathcal{S}$ is defined as a principal value operator, we shall do the computation assuming B to be integrable. The conclusion will then follow by any limit procedure. Following Villani [43], we perform the change of variables $v_* \to v'_*$, for each v, σ fixed. This change of variables is well-defined on $(\cos\theta > 0)$, with Jacobian given by

$$\left|\frac{dv'_*}{dv_*}\right| = \frac{1}{2^N}(1 + k \cdot \sigma) = \frac{(k' \cdot \sigma)^2}{2^{N-1}},$$

where

$$k' = \frac{v - v'_*}{|v - v'_*|}, \quad k' \cdot \sigma = \cos\frac{\theta}{2} > \frac{1}{\sqrt{2}}.$$

Accordingly, we introduce the application $\psi_\sigma : v'_* \longmapsto v_*$, defined on $(k' \cdot \sigma) > 1/\sqrt{2}$. We reproduce here fig. 1 of [43] to describe ψ_σ geometrically. We note that $|v_* - \psi_\sigma(v')| = |v' - v_*|/(k' \cdot \sigma)$, or, what is the same,

$$|v_* - \psi_\sigma(v)| = \frac{|v - v_*|}{k \cdot \sigma}.$$

We apply this change of variable to the part $\int Bf'_*$ in the left-hand side of (46) :

$$\int_{S^{N-1}\times\mathbb{R}^N} d\sigma\, dv_*\, B(v-v_*,\sigma)\, f'_* = \int_{k'\cdot\sigma\geq 1/\sqrt{2}} d\sigma\, dv'_* \left|\frac{dv_*}{dv'_*}\right| B\big(v-\psi_\sigma(v'_*),\sigma\big) f'_*$$

$$= \int_{k'\cdot\sigma\geq 1/\sqrt{2}} d\sigma\, dv'_* \frac{2^{N-1}}{(k'\cdot\sigma)^2}\, B\big(v-\psi_\sigma(v'_*),\sigma\big) f'_*.$$

With our notational convention for the cross-section,

$$B\big(v-\psi_\sigma(v'_*),\sigma\big) = B\big(|v-\psi_\sigma(v'_*)|, k\cdot\sigma\big) = B\Big(|v-\psi_\sigma(v'_*)|, 2(k'\cdot\sigma)^2-1\Big).$$

Changing the name v'_* for v_*, we find that (46) holds with

$$S(|v-v_*|) = \int_{k\cdot\sigma\geq\frac{1}{\sqrt{2}}} d\sigma\, \frac{2^{N-1}}{(k\cdot\sigma)^2} B\Big(|v_*-\psi_\sigma(v)|,\, 2(k\cdot\sigma)^2-1\Big)$$

$$- \int_{k\cdot\sigma\geq 0} d\sigma\, B(|v-v_*|, k\cdot\sigma).$$

The first part is then

$$\int_{k\cdot\sigma\geq\frac{1}{\sqrt{2}}} d\sigma\, \frac{2^{N-1}}{(k\cdot\sigma)^2} B\left(\frac{|v-v_*|}{k\cdot\sigma}, 2(k\cdot\sigma)^2-1\right)$$

$$= |S^{N-2}| \int_0^{\frac{\pi}{4}} d\theta\, \sin^{N-2}\theta\, \frac{2^{N-1}}{\cos^2\theta} B\left(\frac{|v-v_*|}{\cos\theta}, \cos(2\theta)\right)$$

$$= |S^{N-2}| \int_0^{\frac{\pi}{4}} d(2\theta)\, \frac{\sin^{N-2}(2\theta)}{\cos^N\theta} B\left(\frac{|v-v_*|}{\cos\theta}, \cos(2\theta)\right)$$

$$= |S^{N-2}| \int_0^{\frac{\pi}{2}} d\theta\, \frac{\sin^{N-2}\theta}{\cos^N(\theta/2)} B\left(\frac{|v-v_*|}{\cos(\theta/2)}, \cos\theta\right).$$

This proves our first claim. Note the double change of variables ($v'_* \to v_*$, $\theta \to 2\theta$), which is important to recover the right homogeneity.

With the notations of (29) of Assumption I, let us now estimate S. We first consider the contribution S_1 of B_1, i.e. that part of B which is not borderline in (27). Clearly,

$$|S_1(|v-v_*|)| \leq |S^{N-2}| \int_0^{\frac{\pi}{2}} d\theta\, \sin^{N-2}\theta\, \frac{1}{\cos^N(\theta/2)} \left|B_1\left(\frac{|v-v_*|}{\cos(\theta/2)}, \cos\theta\right)\right.$$

$$\left. - B_1(|v-v_*|, \cos\theta)\right|$$

$$+ |S^{N-2}| \int_0^{\frac{\pi}{2}} d\theta\, \sin^{N-2}\theta\, \left[\frac{1}{\cos^N(\theta/2)} - 1\right] B_1(|v-v_*|, \cos\theta).$$

$$\leq |S^{N-2}|\, 2^{N/2} \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \big(1 - \cos(\theta/2)\big)|v - v_*||B_1'(|v - v_*|, \cos\theta)$$

$$+ |S^{N-2}|\, 2^{N/2} \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \big(1 - \cos^N(\theta/2)\big) B_1(|v - v_*|, \cos\theta).$$

Since $1 - \cos^N(\theta/2) \leq N\big(1 - \cos(\theta/2)\big) = 2N\sin^2(\theta/4) \leq N(1 - \cos\theta)/(4\cos^2(\pi/8))$, there is a cancellation of order 2, and we conclude by recalling the definitions of M_1, M_1'.

Now, let us estimate only the borderline contribution, i.e. assume that $B(z, \sigma) = \beta_0(k \cdot \sigma)|z|^{-N}$. Let $B_\varepsilon(z, \sigma) = \beta_0(k \cdot \sigma)|z|^{-N}1_{|z|\geq\varepsilon}$. To this cross-section is associated the convolution kernel

$$S_\varepsilon(|z|) = \frac{|S^{N-2}|}{|z|^N} \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \big[1_{|z|\geq\varepsilon\cos(\theta/2)} - 1_{|z|\geq\varepsilon}\big]\beta_0(\cos\theta)$$

$$= \frac{|S^{N-2}|}{|z|^N} \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \, \beta_0(\cos\theta) 1_{\varepsilon\cos(\theta/2)\leq|z|\leq\varepsilon}.$$

Let

$$I(\delta) = \int_\delta^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta\beta_0(\cos\theta),$$

we have

$$S_\varepsilon(|z|) = \frac{|S^{N-2}|}{|z|^N} I\left(2\cos^{-1}\left(\frac{|z|}{\varepsilon}\right)\right) 1_{|z|\leq\varepsilon}$$

$$= \frac{1}{\varepsilon^N} J\left(\frac{|z|}{\varepsilon}\right),$$

where $J(z) = |S^{N-2}|I(2\cos^{-1}(|z|))|z|^{-N} 1_{|z|\leq 1}$. The fact that the integral of S_ε is constant, and that S_ε is supported in a ball of radius ε easily imply our claim, with $\lambda = \int_{\mathbb{R}^N} J(z)\, dz$, i.e.

$$\lambda = |S^{N-2}||S^{N-1}| \int_0^1 \frac{dr}{r} \int_{2\cos^{-1} r}^{\frac{\pi}{2}} \beta_0(\cos\theta) \, \sin^{N-2}\theta \, d\theta$$

$$= |S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} d\theta \, \sin^{N-2}\theta \, \beta_0(\cos\theta) \int_{\cos(\theta/2)}^1 \frac{dr}{r}$$

$$= -|S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \beta_0(\cos\theta) \log\cos(\theta/2) \, \sin^{N-2}\theta \, d\theta.$$

$$\square$$

Remark : The change of variables $v_* \to v_*'$ introduces a singularity for *frontal* collisions, i.e. those collisions in which $v' \simeq v_*$, $v_*' \simeq v$. We avoid this difficulty by assuming from the beginning, without loss of generality, that B is supported in $(k \cdot \sigma \geq 0)$. We could as well

assume that B is supported in $(k \cdot \sigma \geq -(1 - \delta))$, for some $\delta > 0$, and we would find straightforward modifications of the above results, only with different constants (depending on δ).

3.2. **Dual formulation of the bilinear Boltzmann operator.**

Now, we tackle $(\mathcal{R}_2)$. This term, which is nothing but the action of the bilinear Boltzmann operator on f and $\beta(f)$, essentially involves *fractional derivatives* of these functions. We shall define it in the sense of distributions, by duality. Let $\varphi(v)$ be a (smooth) test-function in the velocity variable, then

$$\int (\mathcal{R}_2)\varphi(v)\,dv = \int_{\mathbb{R}^{2N} \times S^{N-1}} dv\,dv_*\,d\sigma\,B\big[f'_*\beta(f') - f_*\beta(f)\big]\varphi$$
$$= \int_{\mathbb{R}^{2N}} dv\,dv_*\,f_*\beta(f)\left[\int_{S^{N-1}} B(v - v_*, \sigma)(\varphi' - \varphi)\,d\sigma,\right]$$

where we have used the standard change of variables $(\sigma, v, v_*) \leftrightarrow (\sigma, v', v'_*)$, with unit Jacobian. For given v_*, the linear operator

$$(49) \qquad \mathcal{T} : \varphi \longmapsto \int_{S^{N-1}} B(v - v_*, \sigma)(\varphi' - \varphi)\,d\sigma$$

plays a key part in the theory of the Boltzmann equation, to which it is naturally associated, as being the adjoint operator to $f \mapsto Q(\delta_{v_*}, f)$. In the cutoff case, this operator was studied by Lions [35] and Wennberg [47, 48] : it is a compact perturbation of a multiplicative operator. On the other hand, in the non-cutoff case, we are aware of no conclusive study. In [6] it was shown that from the regularity point of view, *averages* of the form $\int dv_*\,g_*\mathcal{T}$ "differentiate at least as much" as a fractional Laplace operator as soon as g has a nonsingular part. We shall show here that $\mathcal{T}$ is essentially bounded as an operator from $W^{2,\infty}$ to L^∞, independently of the strength of the singularity.

Proposition 4 ($W^{2,\infty} \to L^\infty$ bound for $\mathcal{T}$). *Let B satisfy Assumption I. Then, for all $\varphi \in W^{2,\infty}(\mathbb{R}_v^N)$,*

$$|\mathcal{T}\varphi(v)| \leq \frac{1}{2}\|\varphi\|_{W^{2,\infty}}|v - v_*|\left(1 + \frac{|v - v_*|}{2}\right)M(|v - v_*|).$$

Moreover, for all $\alpha \in [0, 2]$ and $\varphi \in W^{2,\infty}(\mathbb{R}_v^N)$,

$$|\mathcal{T}\varphi(v)| \leq 2\|\varphi\|_{W^{2,\infty}}(1 + |v - v_*|)^\alpha\,M^\alpha(|v - v_*|),$$

where M^α is defined by formula (31).

Proposition 4 shows in particular that the cancellation properties exploited in [31, 45] do not require as many symmetries as used there, and hold for the nonsymmetric Boltzmann operator as well as for the symmetric one. In these works, the study of the operator $\mathcal{T}$ is a key step towards the justification of the Landau approximation in the spatially homogeneous case. Before displaying the proof of Proposition 4, we give an immediate corollary :

Corollary 4.1. *Let B satisfy Assumptions I and II, and let f satisfy (45). Then, for all $R > 0$, the term $(\mathcal{R}_2)$ defined by (43) lies in $L^\infty([0,T]; L^1(\mathbb{R}^N_x; W^{-2,1}(B_R(v))))$, where $B_R(v)$ still denotes $\{v \in \mathbb{R}^N, |v| \leq R\}$.*

Proof of Corollary 4.1. For a.a. t, x,

$$\|(\mathcal{R}_2)\|_{W_v^{-2,1}(B_R(v))} = \sup \left\{ \int_{\mathbb{R}^N} (\mathcal{R}_2)\varphi\, dv; \quad \varphi \in W^{2,\infty}(B_R(v)), \ \|\varphi\|_{W^{2,\infty}} \leq 1 \right\}$$

$$\leq \int_{\mathbb{R}^N_x \times B_R(v)} dv\, dv_*\, \beta(f) f_* |\mathcal{T}\varphi|$$

$$\leq \int_{\mathbb{R}^N_x \times B_R(v)} dv\, dv_*\, \beta(f) f_* \big(|v - v_*| M(|v - v_*|) 1_{|v-v_*| \leq R}$$

$$+ 2|v - v_*|^\alpha M^\alpha(|v - v_*|) 1_{|v-v_*| \geq R} \big)$$

$$\leq C(1+R)\|f\|_{L^1_v} \| |z| M(|z|) \|_{L^1(|z| \leq R)} + \varepsilon(R) \left(\int_{\mathbb{R}^N_{v_*} \times B_R(v)} dv_*\, f_* |v - v_*|^2\, dv\, dv_* \right)$$

where we used assumption II. It then suffices to integrate with respect to x. $\qquad\square$

Proof of Proposition 4. By Taylor formula,

$$(50) \quad \varphi(v') - \varphi(v) = (v' - v) \cdot \nabla\varphi(v)$$

$$+ |v' - v|^2 \left[\int_0^1 ds\, (1-s) D^2\varphi\big(v + s(v' - v)\big) \cdot \left(\frac{v' - v}{|v' - v|}, \frac{v' - v}{|v' - v|} \right) \right].$$

By symmetry,

$$\int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma)(v' - v) = \int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma)(v' - v, k)k,$$

where $k = (v - v_*)/|v - v_*|$. But $(v' - v, k)k = -(1/2)|v - v_*|(1 - \cos\theta)k = -(v - v_*)\sin^2(\theta/2)$. Since also $|v' - v|^2 = |v - v_*|^2 \sin^2(\theta/2)$, we find precisely

$$(51) \quad \mathcal{T}\varphi = -\frac{1}{2}M(|v - v_*|)(v - v_*) \cdot \nabla\varphi(v)$$
$$+ \frac{1}{2}|v - v_*|^2 \int_{S^{N-1}} d\sigma\, B(v - v_*, \sigma)(1 - k \cdot \sigma)$$
$$\left[\int_0^1 ds\,(1 - s)D^2\varphi\big(v + s(v' - v)\big) \cdot \left(\frac{v' - v}{|v' - v|}, \frac{v' - v}{|v' - v|}\right)\right].$$

In particular,

$$(52) \quad |\mathcal{T}\varphi| \leq \|\varphi\|_{W^{2,\infty}}\left[\frac{1}{2}M(|v - v_*|)|v - v_*| + \frac{1}{4}|v - v_*|^2 M(|v - v_*|)\right]$$
$$= \frac{1}{2}\left(1 + \frac{|v - v_*|^2}{2}\right)|v - v_*|M(|v - v_*|)\|\varphi\|_{W^{2,\infty}}$$
$$\leq \frac{1}{2}(1 + |v - v_*|^2)M(|v - v_*|)\|\varphi\|_{W^{2,\infty}}.$$

Note that, by the same estimate,

$$|\mathcal{T}\varphi| \leq \left(\frac{1}{2} + \frac{R}{4}\right)\|\varphi\|_{W^{2,\infty}}|v - v_*|M(|v - v_*|) \qquad \text{if } |v - v_*| \leq R.$$

But from the definition of $\mathcal{T}$ it also follows that

$$(53) \qquad\qquad |\mathcal{T}\varphi| \leq 2M^0(|v - v_*|)\|\varphi\|_{L^\infty},$$

and since $M = M^2$, by combining (52) and (53), we obtain the a priori bound

$$|\mathcal{T}\varphi| \leq 2(1 + |v - v_*|^2)^{\alpha/2}M^\alpha(|v - v_*|)\|\varphi\|_{W^{2,\infty}}.$$

$$\square$$

3.3. Integrability of the Γ term.

Finally, we derive an easy a priori estimate for the last term $(\mathcal{R}_3)$, which involves $\Gamma(f, f')$.

Proposition 5. *Let B satisfy assumptions I and II, and let f be a solution of the Boltzmann equation, satisfying (45). Then, for all $R > 0$,*

$$(54) \qquad\qquad (\mathcal{R}_3) \in L^1([0, T]; L^1(\mathbb{R}_x^N \times B_R(v))),$$

where $(\mathcal{R}_3)$ is defined in (44).

Remark : Of course, this Proposition has to be understood in the sense of an a priori estimate, since our goal is precisely to show that the notion introduced in Definition 2, with the help of the renormalized collision operator $\beta'(f)Q(f, f)$, is meaningful.

With Corollaries 3.1, 4.1 and Proposition 5 in hand, we will have – at last ! – shown that our renormalized formulation makes sense.

Proof of Proposition 5. The argument is similar to the one in Lions [38], and relies on the nonnegativity of the integrand in $(\mathcal{R}_3)$. Let us integrate equation (21), in the form

$$\frac{\partial \beta(f)}{\partial t} + v \cdot \nabla \beta(f) \geq (\mathcal{R}_1) + (\mathcal{R}_2) + (\mathcal{R}_3),$$

in all variables, against a test-function $\varphi(v) \geq 0$, $\varphi \equiv 1$ on $B_R(v)$, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2R}(v)$, $\|\varphi\|_{W^{2,\infty}} \leq CR^{-2}$. Thanks to our assumptions on β, we have $\beta(f) \leq Cf$, so that

$$\int_{\mathbb{R}_x^N \times B_{2R}(v)} \beta(f(T,x,v)) \, dx \, dv \leq C\|f_0\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)}.$$

Also $\int v \cdot \nabla_x \beta(f) \varphi(v) \, dv \, dx = 0$ (this computation is easily justified by approximation, using the bounds (45) which ensure decay of f at infinity). Also the integrals of $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$ are bounded according to Corollaries 3.1 and 4.1. Since $(\mathcal{R}_3)$ is nonnegative, we are left with (54). $\qquad\square$

4. STRONG COMPACTNESS AND PASSAGE TO THE LIMIT

In this section, we prove Theorem 2. We shall not completely detail the argument below, because some parts of it are quite similar to existing proofs in [23, 34, 35, 38, 39]. So we only insist on the new features of the proof.

We proceed by approximation. There are several ways of doing this, and one possible would be to choose approximate equations in the same spirit as DiPerna and Lions, for instance

$$(55) \qquad \frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n = \frac{\widetilde{Q}_n(f^n, f^n)}{1 + \frac{1}{n}\|f^n\|_{L_v^1}},$$

where $\|f\|_{L_v^1}(t,x) = \int f(t,x,v) \, dv$, and $\widetilde{Q}_n$ is a Boltzmann operator with a suitable mollification B_n of the cross-section B. Also the initial datum f_0 should be approximated by a smooth function f_0^n with rapid decay, and with a bound below of the form $C_n e^{-\delta_n(|v|^2 + |x|^2)}$, so that formal manipulations involving the logarithm are all admissible... Yet it will be simpler, and more satisfactory, to start from the known results of DiPerna and Lions, of existence in the cutoff case, and thus to deal only with solutions of true Boltzmann equations, where the only approximation will be performed at the level of the cross-section.

Definition 3. *Let* $(B_n)_{n \in \mathbb{N}} \cup \{B\}$ *be a sequence of cross-sections satisfying Assumptions I and II. We denote quantities attached to each* B_n *as in* (24),(47), *by* M_n, S_n, *etc. We shall say that* B_n *approximates* B *(and write* $B_n \longrightarrow B$ *) if*

(i) $S_n \longrightarrow S$, *locally in weak-measure sense;*

(ii) $\mathcal{T}_n \longrightarrow \mathcal{T}$, *in weak (distributional) sense;*

(iii) $B_n \longrightarrow B$ *a.e. on* $\mathbb{R}^N \times S^{N-1}$;

(iv) As $|z| \to \infty$, $M_n^\alpha = o(|z|^{2-\alpha})$ *for some* $\alpha \in [0,2]$, *and* $|z| M_n'(|z|) = o(|z|^2)$, *uniformly in* n.

Remark : If $|z| M_n(|z|) \to |z| M(|z|)$ (locally in weak-measure sense) and

$$|z|^{2-\epsilon} B^n(z, \sigma)(1 - k \cdot \sigma)^{1-\delta} \longrightarrow |z|^{2-\epsilon} B(z, \sigma)(1 - k \cdot \sigma)^{1-\delta}$$

for some $\epsilon, \delta > 0$, then $\mathcal{T}_n \longrightarrow \mathcal{T}$ weakly, according to formula (51) (note that for any $\epsilon, \delta > 0$, the function $|v - v_*|^\epsilon (1 - k \cdot \sigma)^\delta (v' - v)/|v' - v|$ is a continuous function of both σ and $v - v_*$).

Example : If B satisfies assumption I, the most simple way to approximate B is of course to choose $B_n(z, \sigma) = B(z, \sigma) 1_{|z| \geq 1/n, \theta \geq 1/n}$.

We also need a condition to express the fact that "on the whole", the sequence (B_n) is singular enough :

Assumption III' *(Overall angular singularity condition). We require that for all* n,

$$(56) \qquad B_n(z, \sigma) \geq \Phi_0(|z|) b_{0,n}(k \cdot \sigma), \qquad k = \frac{z}{|z|},$$

for some fixed continuous function $\Phi_0(|z|)$ *such that* $\Phi_0(|z|) > 0$ *if* $z \neq 0$, *and*

$$(57) \qquad \int_{S^{N-1}} \varliminf_{n \to \infty} b_{0,n}(k \cdot \sigma) \, d\sigma = +\infty.$$

Remark : Of course, if B satisfies Assumption III, then without loss of generality we may choose the same function Φ_0 in Assumptions III and III'.

Taking into account the results of DiPerna and Lions, Theorem 2 will be a byproduct of the following extended stability theorem.

Theorem 6 (Extended stability). *Let* B *satisfy Assumptions I, II, III, and let* $(B_n)_{n \in \mathbb{N}}$ *be a sequence of cross-sections such that* $B_n \longrightarrow B$. *Assume that this sequence satisfies the overall singularity condition III'. Let* $(f^n)_{n \in \mathbb{N}}$ *be a sequence of solutions to the Boltzmann equation with*

respective cross-section B_n, in the sense of Definition 2. Assume that the sequence (f^n) satisfies the natural a priori bounds (16), (18) (with B replaced by B_n, of course). Assume without loss of generality that

$$f^n \longrightarrow f \quad weakly \ in \ L^p([0,T], L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)) \ (1 \le p < +\infty).$$

Then

(i) f^n converges strongly towards f;

(ii) f is a renormalized solution with a defect measure of the Boltz-mann equation with cross-section B;

(iii) f satisfies (36)–(39).

Remark : The control on $\int f^n \log f^n$ and $\int f^n(1 + |v|^2 + |x|^2) \, dx \, dv$ also implies classicaly control on $\int f^n |\log f^n|$, and by Dunford-Pettis Theorem, this implies that the sequence (f^n) is uniformly equicontin-uous, and lies in a weakly compact set of L^1. This is why, extracting subsequences if necessary, we may assume that

$$f^n \longrightarrow f \qquad in \ w - L^p([0,T], L^1(\mathbb{R}_v^N \times \mathbb{R}_x^N)), \qquad 1 \le p < \infty.$$

Proof of Theorem 6. We first prove that the convergence is strong. We shall follow the general strategy exposed in Lions [38]. Once this is done, it will remain to pass to the limit in the equation as $n \to \infty$.

1) The first step is to write the renormalized formulation, that is,

$$\frac{\partial \beta(f^n)}{\partial t} + v \cdot \nabla_x \beta(f^n) = \beta'(f^n) Q(f^n, f^n) + \mu^n,$$

where, for all $R > 0$, μ^n is a nonnegative measure, with finite mass on $[0,T] \times \mathbb{R}_x^N \times B_R(v)$. According to all the bounds proven in section 3,

$$\frac{\partial \beta(f^n)}{\partial t} + v \cdot \nabla_x \beta(f^n) = g^n + \sum_i \frac{\partial}{\partial v_i} g_i^n + \sum_{ij} \frac{\partial^2}{\partial v_i \partial v_j} g_{ij}^n,$$

where g^n, g_i^n, g_{ij}^n are locally integrable, or locally bounded measures. This entails, by the so-called averaging lemmas [29, 26, 34], that aver-ages of the form $\int \beta(f^n) \varphi(v) \, dv$, for $\varphi \in C_0^\infty(\mathbb{R}^N)$, are strongly com-pact in the variables (t, x). Reasoning as in [23, 38], and using a priori bounds on f^n, one deduces from this that convolution products of the form $\beta(f^n) *_v \varphi$ are also strongly compact in $L^1([0,T] \times \mathbb{R}_x^N \times \mathbb{R}_v^N)$.

Reasoning as in [23, 38] (replacing $\beta(f^n)$ by a concave approximation of the square root function and using the a priori bounds (16) together with standard arguments from functional analysis), this also implies that convolution products of the form $\sqrt{f^n} *_v \varphi$ are strongly compact in L^1. This result is a somewhat general consequence of the existence of a renormalized formulation.

Another useful information, obtained by combining the averaging lemmas, the renormalized formulation and the bounds on (f^n), is that $\int f^n \, dv$ is strongly compact in (t, x), and thus converges strongly to $\int f \, dv$.

2) The next and most delicate step is to use a smoothness estimate in the velocity variable. It will be a consequence of the entropy dissipation estimate and the overall singularity assumption III'. At this point we invoke the results that we have established together with Desvillettes and Wennberg in [6]. Let

$$Z_n(a) = |S^{N-2}| \int_a^{\frac{\pi}{2}} b_{0,n}(\cos\theta) \sin^{N-2}\theta \, d\theta,$$

and let $f_R^n = f^n \chi_R$, where $\chi(v)$ is a smooth cutoff function with support in $B_{3R}(v)$, identically equal to 1 on $B_{2R}(v)$.

In [6] we have proven the following result : for all $0 < L < +\infty$, $0 < \varepsilon < +\infty$, for a.a. (t, x) such that

$$H^n(t, x) \equiv \int f^n(t, x, v)\big(1 + |v|^2 + \log f^n(t, x, v)\big) \, dv \leq L < +\infty,$$

and

$$\rho^n(t, x) \equiv \int f^n(t, x, v) \, dv \geq \varepsilon > 0,$$

the following *pointwise* estimate in (t, x) holds :

$$(58) \quad \int_{|\xi| \geq 1} |\mathcal{F}\sqrt{f_R^n}|^2 Z_n\left(\frac{1}{|\xi|}\right) d\xi \leq C(f^n, R, \Phi_0)\Big[D(f^n) + \|f^n\|_{L_2^1}^2\Big].$$

Here $\mathcal{F}$ denotes the usual Fourier transform with respect to the velocity variable, and $C(f^n, R, \Phi_0)$ is a constant depending only on R, Φ_0, and on L, ε.

Thus (loosely speaking), taking into account the bounds

$$\sup_n \sup_t \int_{\mathbb{R}^N} dx \, D(f^n) < +\infty, \quad \sup_n \sup_t \int_{\mathbb{R}^N} dx \, H^n(t, x) < +\infty,$$

we see that the entropy dissipation estimate implies a smoothness estimate for $\sqrt{f^n}$ in the velocity variable, out of

(i) a set of small measure in (t, x) where f^n may have infinite mass, energy or entropy, and

(ii) a set where the density $\rho^n = \int f^n \, dv$ may be very small.

A precise formulation is easy, in the same spirit as in Lions [38, 39]. Let us fix a large number R. By the a priori bounds on f, it is clear that

$g^n = \sqrt{f^n} \rightharpoonup g$ for some function $g \in L^2$. To prove strong convergence, it is enough to prove that $g^n \to g$ a.e. on each $W_\varepsilon \times B_R(v)$, where

$$W_\varepsilon = \left\{ (t,x); \quad t \in [0,T], \ |x| < R, \ \int g^2 \, dv > \varepsilon \right\}.$$

Indeed, note that $g^n \to g = 0$ in $L^1(|x| < R, \ |v| < R, \ \int g^2 \, dv = 0)$ (a sequence of nonnegative functions, converging to 0 weakly in L^1, automatically converges to 0 strongly in L^1).

Now, on W_ε, by convexity of the square function,

$$\varepsilon < \int g^2 \, dv \leq \varliminf_{n \to \infty} \int (g^n)^2 \, dv = \varliminf_{n \to \infty} \int f^n \, dv = \int f \, dv$$

where we have used the fact that $\int f^n \, dv \longrightarrow \int f \, dv$ strongly. By Egorov's theorem, for all $\delta > 0$ there is a Borel set U_δ, with measure less than δ, such that $\int f^n \, dv \longrightarrow \int f \, dv$ uniformly on $W_\varepsilon \setminus U_\delta$. For n large enough, this implies that $\int f^n \, dv \geq \varepsilon/2$ on $W_\varepsilon \setminus U_\delta$.

Next, let

$$V_L^n = \{(t,x); \ H_n(t,x) > L\}.$$

Our bounds imply that $|V_L^n| \leq CL^{-1}$, where C is a constant independent on n. And our entropy dissipation estimate implies that there is another constant C such that for all $A \geq 1$,

$$\int_{W^\varepsilon \setminus (U_\delta \cup V_L^n)} dt\, dx \int_{|\xi| \geq A} d\xi \, |\mathcal{F}\sqrt{f_R^n}|^2 \leq \frac{C}{Z_n\left(\frac{1}{A}\right)}.$$

Passing to the lim sup on both sides, we find

$$\varlimsup_{n \to \infty} \int_{W^\varepsilon \setminus (U_\delta \cup V_L^n)} dt\, dx \int_{|\xi| \geq A} d\xi \, |\mathcal{F}\sqrt{f_R^n}|^2 \leq \frac{C}{Z_\infty\left(\frac{1}{A}\right)},$$

where

$$Z_\infty(a) = \varliminf_{n \to \infty} |S^{N-2}| \int_a^{\frac{\pi}{2}} b_{0,n}(\cos\theta) \, \sin^{N-2}\theta \, d\theta$$

$$\geq |S^{N-2}| \int_a^{\frac{\pi}{2}} \varliminf_{n \to \infty} b_{0,n}(\cos\theta) \, \sin^{N-2}\theta \, d\theta.$$

From formula (57) we know that

$$Z_\infty(a) \xrightarrow{a \to 0} +\infty.$$

Thus, for each L, ε, δ, we have

$$\lim_{A \to \infty} \varlimsup_{n \to \infty} \int_{W^\varepsilon \setminus (U_\delta \cup V_L^n)} dt\, dx \int_{|\xi| \geq A} d\xi \, |\mathcal{F}\sqrt{f_R^n}|^2 = 0.$$

On the other hand, since $|U_\delta \cup V_L^n| \le \delta + L^{-1}$, and since (f^n) is uniformly equi-integrable,

$$\lim_{\substack{\delta \to 0 \\ L \to +\infty}} \sup_{n \in \mathbb{N}} \int_{U_\delta \cup V_L^n} dt\, dx \int_{\mathbb{R}^N} d\xi\, |\mathcal{F}\sqrt{f_R^n}|^2 = 0.$$

On the whole,

$$(59) \qquad \lim_{A \to +\infty} \overline{\lim_{n \to \infty}} \int_{W^\varepsilon} dt\, dx \int_{|\xi| \ge A} d\xi\, |\mathcal{F}\sqrt{f_R^n}|^2 = 0.$$

This is the "velocity smoothness" estimate that we needed.

3) Now, let $\rho_\delta = \delta^{-N}\rho(\cdot/\delta)$, $\delta > 0$, be a family of mollifiers in the velocity variable (ρ smooth, nonnegative, compactly supported, $\int \rho = 1$). From (59) follows

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \|\sqrt{f_R^n} - \sqrt{f_R^n} * \rho_\delta\|_{L^2(W_\varepsilon \times B_R(v))} = 0.$$

Clearly, in view of our truncation procedure this is the same as

$$\lim_{\delta \to 0} \overline{\lim_{n \to \infty}} \|\sqrt{f^n} - \sqrt{f^n} * \rho_\delta\|_{L^2(W_\varepsilon \times B_R(v))} = 0.$$

Since, for any $\delta > 0$, $\sqrt{f^n} * \rho_\delta$ lies in a strongly compact set in L^2, this entails that $(\sqrt{f^n})$ is also relatively strongly compact.

4) We can now prove strong compactness as in Lions [38]. From the strong compactness of $(\sqrt{f^n})$ follows the a.e. convergence of (f^n). Since (f^n) already converges weakly towards f, this entails that

$$f^n \longrightarrow f \qquad \text{in } L^1([0, T] \times R_x^N \times R_v^N).$$

5) Once the strong convergence has been established, it is easy to pass to the limit in the renormalized formulation with a defect measure. First, it is clear that

$$\frac{\partial \beta(f^n)}{\partial t} \longrightarrow \frac{\partial \beta(f)}{\partial t}, \qquad v \cdot \nabla_x \beta(f^n) \longrightarrow v \cdot \nabla_x \beta(f)$$

in weak sense.

Then, we handle the term $(\mathcal{R}_1)$. With the notations of Proposition 3, for any test-function φ (smooth with compact support),

$$\int (\mathcal{R}_1)^n \varphi = \int_{\mathbb{R}^{2N}} [f^n \beta'(f^n) - \beta(f^n)] f_*^n S^n(v - v_*)\varphi(v)\, dv\, dv_*.$$

Since $(f^n \beta'(f^n) - \beta(f^n))$ is relatively compact in weak-* L^∞, it suffices to show that $(f^n * S^n)$ is relatively strongly compact in L^1. This follows immediately from the bounds on S^n and f^n at infinity, the fact that S^n

converges locally in weak-measure sense, and the strong convergence of f^n.

Next, we consider $(\mathcal{R}_2)$. For a sufficiently smooth test-function φ, we have

$$\int (\mathcal{R}_2)^n \varphi \, dv = \int_{\mathbb{R}^{2N}} f_*^n \beta(f^n)(\mathcal{T}^n \varphi) \, dv \, dv_*,$$

where $\mathcal{T}^n$ is the linear operator defined in (49). Thus, it is immediate that

$$\int (\mathcal{R}_2)^n \varphi \, dv \, dx \longrightarrow \int (\mathcal{R}_2) \varphi \, dv \, dx,$$

since $\mathcal{T}^n \to \mathcal{T}$ in weak sense, as an operator on $L^1(\mathbb{R}_v^N \times \mathbb{R}_{v_*}^N)$, and $f_*^n \beta(f^n) \to f_* \beta(f)$ in strong L^1 sense, locally on $\mathbb{R}_x^N \times \mathbb{R}_{v_*}^N \times \mathbb{R}_v^N$.

Finally, $(\mathcal{R}_3)^n$ is bounded in measure sense, locally in velocity space, so that up to extraction, it converges weakly to some nonnegative limit. Let $\varphi(t, x, v)$ be a nonnegative test-function; since $f^n \longrightarrow f$ and $B_n \longrightarrow B$ a.e., it follows at once from Fatou's lemma that

$$\int (\mathcal{R}_3) \varphi \leq \varliminf_{n \to \infty} \int (\mathcal{R}_3)^n \varphi,$$

which proves that, in weak sense, $(\mathcal{R}_3) \leq \varliminf(\mathcal{R}_3)^n$.

6) As for the mass and momentum conservation laws (36) and (37), they are easy consequences of the convergence of f^n towards f and the uniform energy bounds, while the identities (38) and (39) are implied by Fatou's lemma again.

$\square$

Remark : Under suitable assumptions on the cross-section, this proof of strong compactness may possibly be turned into explicit smoothness bounds by the strategy of Desvillettes and Golse [21].

5. Is a borderline kinetic singularity regularizing ?

Our renormalized formulation covers such cross-sections as

$$(60) \qquad \sin \theta \, B_n(|z|, \cos \theta) = \frac{1}{\log n} \frac{1}{|z|^3} \frac{\cos(\theta/2)}{\sin^3(\theta/2)} 1_{\theta \geq \frac{1}{n}}$$

in dimension $N = 3$, which is sometimes used as an approximation of the cross-section associated to the screened Debye potential (see [15] for instance). But we are unable to prove stability/existence results for such a cross-section because (for a given n) it is *nonsingular in θ*, so that the singularity condition III does not hold.

We insist that this is an artifact of the approximation. Like any cross-section associated to a long-range potential, the Debye cross-section *is*

singular in the angular variable θ, as we shall check in [7]. Ironically, this approximation, which is performed to get rid of the difficulties caused by too strong an angular singularity, yields too weak an angular singularity for our theorem to apply !

On the other hand, the kernel is nonintegrable in the relative velocity variable, and we may conjecture that this is sufficient to entail strong compactness. Our conjecture relies on the following observation. Assume that

$$B(z, \sigma) \geq \Phi_0(|z|) b_0(k \cdot \sigma), \ \ k = z/|z|,$$

where $\Phi_0(|z|) \geq K|z|^{-N}$ and b_0 is not identically zero. Assume without loss of generality that b_0 is identically vanishing close to $\theta = 0$, and replace it by its symmetric version. Since we have cut frontal collisions, we are then allowed to use the Cancellation Lemma, even for a borderline singularity.

Starting from the entropy dissipation estimate and applying the same method as in [7], we can prove

$$\int dt \, dx \, dv \, dv_* \, d\sigma \, b_0(k \cdot \sigma) f_* \frac{(\beta(f) - \beta(f'))^2}{|v - v'|^N} < +\infty.$$

This inequality can be transformed by using the symmetries

$$(v, v_*) \leftrightarrow (v_*, v), \ (v', v'_*, k) \to (v, v_*, \sigma), \ \sigma \to -\sigma$$

(which may more easily be thought of as $k \to -k$, $\sigma \leftrightarrow k$, $\sigma \to -\sigma$). Combining the resulting estimates with the elementary inequalities

$$\left[\beta(f) - \beta(f_*)\right]^2 \leq 2\left[\beta(f) - \beta(f')\right]^2 + 2\left[\beta(f') - \beta(f_*)\right]^2$$

and $|v - v'| \leq |v - v_*|$, one can prove that
(61)
$$\int dt \, dx \, dv \, dv_* \left[\int_{S^{N-1}} d\sigma \, b_0(k \cdot \sigma)[f(v') + f(v'_*)]\right] \frac{(\beta(f) - \beta(f_*))^2}{|v - v_*|^N} < +\infty.$$

If the coefficient in square brackets was locally bounded below, this would entail a control on $\beta(f)$ locally in

$$\log H = \left\{ F \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} |\widehat{F}(\xi)|^2 \log(1 + |\xi|) \, d\xi < +\infty \right\}.$$

Since the coefficient $\int_{S^{N-1}} d\sigma \, b_0(k \cdot \sigma) f(v')$ is an integral over a manifold of codimension 1, and therefore may take arbitrarily small values unless f is locally bounded below, we unfortunately cannot conclude the argument and prove strong compactness. In fact, it seems extremely intricate to exclude the possibility that f may take very large

and very small values at very close points. This difficulty, which is of genuinely nonlinear nature, is the same that appeared in Villani [43].

One striking aspect is that the estimate (61) above is, in some sense, geometrically dual to the estimate extracted from [43] in the case of a logarithmic angular singularity (i.e. when $\sin^{N-2} b_0(\cos\theta) \geq K\theta^{-1}$). This last estimate reads

$$\int dt\, dx\, dv\, dv' \left(\int_{E_{vv'}} dv_* \, \Phi_0(|v'-v'_*|)|v'-v'_*|[f(v_*)+f(v'_*)] \right)$$
$$\frac{\left[\beta(f)-\beta(f')\right]^2}{|v-v'|^N} < +\infty,$$

where $E_{vv'}$ is the hyperplane orthogonal to $[v, v']$, going through v.

In the case of an angular singularity, the use of Fourier analysis, performed in [6], allowed the removal of the difficulties linked to the possible vanishing of f, and eventually led to the proof of the local smoothness in $\log H$. But now we are discussing the case of a *kinetic* singularity, and the situation is much more complicated because the Fourier transform behaves nicely as regards the angular part of the cross-section, but not the kinetic part. One could wonder whether a method similar to the one used by Lions [39] may work in that case.

If this conjecture was false, this would mean that such approximations as (60) are certainly non relevant from the physical point of view, since they would not enjoy the important compactifying property of the Debye potential.

Appendix : On the defect measure

In spite of our remarks on our notion of weak solutions, the reader may legitimately ask how far we are from being able to build a theory of renormalized solutions in the usual sense, or equivalently to prove that the defect measure automatically vanishes. In this long appendix, we discuss a strategy for such a result, and the main difficulties which arise in its tentative implementation.

First of all, let us review existing papers where similar problems are treated. The first one is the work [22] by DiPerna and Lions, where the Boltzmann operator *with cut-off* is perturbed by a linear diffusion operator Δ_v. In a renormalization procedure, the diffusive term a priori leads to a defect measure. But DiPerna and Lions are able to show that the entropy dissipation bound is sufficient to prove its vanishing. In [44] it was also shown that this is formally true for the Landau equation, but the manipulations were too complicated to give hope of rigorous

justification. A second family of works where this problem is treated is the series of papers initiated by Blanchard, Murat and co-workers, on renormalized parabolic equations (see [11, 12] and the numerous works cited therein).

By comparison with all these works, the first serious difficulty in our problem is that *we do not have a good smoothness estimate for truncations of the solution.* The best a priori smoothness estimate at our disposal for the Boltzmann equation without cut-off is given by the entropy dissipation. By the same arguments as discussed in [7], it is possible to prove from the entropy dissipation estimate the bound

$$(62) \qquad \int B(f_* + f_*')[\sqrt{f} - \sqrt{f'}]^2 - \int (\mathcal{S}f)f < +\infty.$$

But we cannot exclude the possibility that both integrals in (62) be infinite. Apparently, the best available estimate is

$$(63) \qquad \int B(f_* + f_*')[\beta(f) - \beta(f')]^2 < +\infty$$

for, say $\beta(f) = f/(1 + f)$. Assume, to fix the ideas, that the cross-section has an angular singularity of order $1 + \nu$, $\nu > 0$. Then from (63) we can deduce

$$\beta(f) \in L_{t,x}^2 \left(A_{L,\varepsilon}; H_{\text{loc}}^{\nu/2}(\mathbb{R}_v^N) \right),$$

where

$$A_{L,\varepsilon} = \left\{ (t,x); \quad \int f \, dv \geq \varepsilon, \quad \int f(1 + |\log f| + |v|^2) \, dv \leq L \right\}.$$

Thus the smoothness estimate for $\beta(f)$ holds only on a rough set in (t,x).

On the other hand, in the Fokker-Planck-Boltzmann case, the entropy dissipation yields the estimate

$$\frac{|\nabla f|^2}{f} \in L^1(\mathbb{R}^+ \times \mathbb{R}_x^N \times \mathbb{R}_v^N),$$

or equivalently $\sqrt{f} \in L_{t,x}^2(H_v^1)$. Also in [12] and related works, an estimate like $\beta(f) \in L_t^p(W_x^{1,p})$ is available. Thus the situation here looks much worse...

Let us continue our discussion. We think that two phenomena are strongly underlying the arguments of the aforementioned works, and in particular of DiPerna and Lions [22] :

1) For a well-chosen family of nonlinearities (β_α), approaching the identity as $\alpha \to 0$, the corresponding defect measure $\mu_\alpha := \mu_{[\beta_\alpha]}$ goes to 0;

2) Using a bit of smoothness, one can prove that μ_α is *nondecreasing* as α decreases to 0.

Combination of points 1) and 2) implies of course that μ_α is vanishing for all α.

Let us discuss first point 1). The simplest way to prove it is by direct control at the level of the sequence of approximate solutions. This is the approach used in [22] : for a given nonlinearity β, used in the renormalized formulation, the authors construct a family $\beta_\alpha = \Psi_\alpha \circ \beta$ of other admisssible nonlinearities, such that

$$(64) \qquad -[\Psi_\alpha \circ \beta]''(f) \le \frac{\varepsilon(\alpha)}{f}, \qquad \varepsilon(\alpha) \xrightarrow[\alpha \to 0]{} 0$$

(α corresponds to the couple (θ, R) in the notations of [22]). Combining (64) with the trivial estimate

$$\int d\mu_{[\beta_\alpha]} \le \overline{\lim_{n \to \infty}} \int -(\Psi_\alpha \circ \beta)''(f^n)|\nabla f^n|^2,$$

and with the entropy dissipation bound

$$(65) \qquad \sup_n \int \frac{|\nabla f^n|^2}{f^n} < +\infty,$$

where the integral is taken over all of $[0, T] \times \mathbb{R}_x^N \times \mathbb{R}_v^N$, it becomes clear that

$$\int d\mu_{[\beta_\alpha]} \xrightarrow[\alpha \to 0]{} 0.$$

In our case, such a control is not available. Indeed, roughly speaking, taking into account our worse entropy dissipation estimate (63), we would need to find something like

$$(66) \qquad -(\Psi_\alpha \circ \beta)''(f) \le \frac{\varepsilon(\alpha)}{(1+f)^2}.$$

This is plainly impossible ! Indeed, assuming $(\Psi_\alpha \circ \beta)'(0) \simeq 1$, $(\Psi_\alpha \circ \beta)'(+\infty) \simeq 0$ as $\alpha \to 0$, we see that $\int -(\Psi_\alpha \circ \beta)''(f)\, df$ is of order 1. Thus, what makes (64) possible and (66) impossible is the fact that $\int_0^{+\infty} df/f = +\infty$, while $\int_0^{+\infty} df/(1+f)^2 < +\infty$.

There is another way towards point 1) above, exploited in [12] and related works. Recall that our renormalized equation is of the form

$$\frac{\partial \beta_\alpha(f)}{\partial t} + v \cdot \nabla_x \beta_\alpha(f) = \beta_\alpha'(f)Q(f, f) + \mu_\alpha$$

$$= (\mathcal{R}_1)_\alpha + (\mathcal{R}_2)_\alpha + (\mathcal{R}_3)_\alpha + \mu_\alpha.$$

Let us integrate this equation in all variables on $[0, T] \times \mathbb{R}_x^N \times \mathbb{R}_v^N$. We should actually first multiply by a cut-off function with compact

support, and then pass to the limit, but this is not a problem. Recalling that $\int (\mathcal{R}_2) = 0$ and $(\mathcal{R}_3) \geq 0$, we obtain

$$\int d\mu_\alpha \leq \int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} \beta_\alpha(f(T, \cdot, \cdot)) \, dx \, dv - \int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} \beta_\alpha(f_0) \, dx \, dv$$
$$+ \left| \int (\mathcal{R}_1)_\alpha \, dt \, dx \, dv \right|.$$

Since β_α approaches the identity as $\alpha \to 0$, in view of the mass conservation we have $\int \beta_\alpha(T, \cdot, \cdot) - \int \beta_\alpha(f_0) \longrightarrow 0$ as $\alpha \to 0$. Thus $\int d\mu_\alpha \longrightarrow 0$ as soon as $\int (\mathcal{R}_1)_\alpha \longrightarrow 0$, i.e

$$(67) \qquad \left| \int [f \beta'_\alpha(f) - \beta_\alpha(f)] \mathcal{S} f \right| \xrightarrow[\alpha \to 0]{} 0.$$

Under what conditions can such an identity hold true ? In the case $\beta_\alpha(f) = f/(1 + \alpha f)$, we cannot conclude to (67) under a plain L^1 estimate. Actually, an L^2 *estimate for f* in all variables would be sufficient, since in that case

$$\beta_\alpha(f) - f \beta'_\alpha(f) = \frac{\alpha f^2}{(1 + \delta f)^2} \leq \left[\sup_{y \in \mathbb{R}^+} \frac{y}{(1 + y^2)} \right] f,$$

and $f \mathcal{S} f$ lies in L^1 if $f(1 + |v|^2) \in L^1$, $f \in L^2$. Thus invoking Lebesgue's dominated convergence theorem would imply the result.

Even with other choices of nonlinearities, we have been unable to do better than this L^2 condition. As a conclusion, we see that point 1) above seems out of reach. This is also a consequence of the problem of large cancellations, since the bound we would need is precisely finiteness of the second integral in (62).

Let us now discuss point 2), which is subtler (but also inaccessible so far !) The main idea behind point 2), as used in [22], is that the family $\beta_\alpha = \Psi_\alpha \circ \beta$ is constructed in such a way that

$$(68) \qquad 1 \leq \Psi'_\alpha \leq C_\alpha$$

The lower bound is natural : starting from a sublinear concave nonlinearity, it is natural to construct better approximations of the identity by *increasing* the derivative. And the fact that $\Psi'_\alpha \geq 1$ formally implies that

$$(69) \qquad \mu_{[\Psi_\alpha \circ \beta]} \geq \mu_{[\beta]}.$$

Indeed, if we multiply the equation for $\beta(f)$ by $\Psi'_\alpha(\beta(f))$, we find, using the chain-rule,

$$\frac{\partial}{\partial t}[\Psi_\alpha \circ \beta(f)] + v \cdot \nabla_x[\Psi_\alpha \circ \beta(f)] = (\Psi_\alpha \circ \beta)'(f)Q(f,f) + \Psi'_\alpha(\beta(f))\mu_{[\beta]}$$
$$\geq (\Psi_\alpha \circ \beta)'(f)Q(f,f) + \mu_{[\beta]}.$$

But, by definition of the defect measure,

$$\frac{\partial}{\partial t}[\Psi_\alpha \circ \beta(f)] + v \cdot \nabla_x[\Psi_\alpha \circ \beta(f)] = (\Psi_\alpha \circ \beta)'(f)Q(f,f) + \mu_{[\Psi_\alpha \circ \beta]},$$

so that (69) holds. To make this argument perfectly correct, it would be necessary to give a precise definition of $\Psi'_\alpha(\beta(f))\,d\mu_\alpha$ — but whatever the definition, it is reasonable to assume that this quantity is greater than $d\mu_\alpha$ in distributional sense.

Thus we see that point 2) above actually amounts to a justification of the *chain-rule for renormalized solutions*; more precisely, a justification of the formal equality

(70) $$\Psi'(\beta(f)) \times [\beta'(f)Q(f,f)] = (\Psi \circ \beta)'(f)Q(f,f)$$

(from now on, we drop the index α for Ψ, since we just need to work for a fixed index).

Let us discuss this point more in detail in the case of the Boltzmann equation without cut-off. First of all, there are two possible ways of rigorously defining the right-hand side of (70) as a distribution. The first is to use our renormalized formulation with nonlinearity $\Psi \circ \beta$. In doing so, we assume that $\Psi \circ \beta$ satisfies all the necessary requirements – in particular, concavity.

The second way is to multiply the definition of $\beta'(f)Q(f,f)$ by the function $\Psi'(\beta(f))$. How to give sense to this one ?

a) $\Psi'(\beta(f))(\mathcal{R}_1)$ is well-defined as $\Psi'(\beta(f))[f\beta'(f) - \beta(f)](\mathcal{S}f)$;

b) $\Psi'(\beta(f))(\mathcal{R}_3)$ is still well-defined as a nonnegative L^1_{loc} function.

c) It remains to give sense to $\Psi'(\beta(f))Q(f,\beta(f))$. This is easy if we recall our general *asymmetric* renormalized formulation, with $\beta(f)$ in place of f ! This will give rise to three terms denoted by $(\widetilde{\mathcal{R}}_1)$, $(\widetilde{\mathcal{R}}_2)$, $(\widetilde{\mathcal{R}}_3)$.

This second definition is nothing but an *iterated renormalization* procedure. Formal identification of both formulas for $\Psi'(\beta(f))\beta'(f)Q(f,f)$ leads to the two identities

$$\Psi'(\beta(f))[f\beta'(f) - \beta(f)] + [\beta(f)\Psi'(\beta(f)) - \Psi(\beta(f))]$$
$$= f(\Psi \circ \beta)'(f) - \Psi \circ \beta(f),$$

$$\Gamma_{\Psi \circ \beta}(f, f') = \Psi'(\beta(f))\Gamma_\beta(f, f') + \Gamma_\Psi\big(\beta(f), \beta(f')\big),$$

valid for all nonnegative numbers f, f'. These identities, which actually reduce to the usual chain-rule, ensure that our two different definitions of $\Psi'(\beta(f))\beta'(f)Q(g, f)$ coincide.

If we could multiply the renormalized equation by $\Psi'(\beta(f))$ and recover the iterated renormalization formulation, we would be done. But the problem is that there are two different ways of seeing the term $(\mathcal{R}_2) = Q(f, \beta(f))$:

- by duality, acting on $\mathcal{D}'$;

- or in renormalized framework, in the definition of $\Psi'(\beta(f))Q(f, \beta(f))$.

One can see the regularization argument in [22] as a way to solve this problem : introduce a family of mollifiers (ρ_ε) in variables t, x, v, defined through the usual procedure, convolve the equation in renormalized form by ρ_ε and multiply by $\Psi'(\beta(f) * \rho_\varepsilon)$. Then, what comes out is

$$\frac{\partial}{\partial t}\Psi(\beta(f) * \rho_\varepsilon) + [v \cdot \nabla_x \Psi(\beta(f))] * \rho_\varepsilon \, \Psi'(\beta(f) * \rho_\varepsilon)$$
$$\geq \Psi'(\beta(f) * \rho_\varepsilon)[\beta'(f)Q(f, f)] * \rho_\varepsilon + \mu * \rho_\varepsilon,$$

where we have used again $\Psi' \geq 1$. We can pass to the limit as $\varepsilon \to 0$ in the left-hand side (transport term), as shown in [22] by a simple argument. Also $\mu * \rho_\varepsilon \longrightarrow \mu$ weakly, and $\Psi'(\beta(f) * \rho_\varepsilon)(\mathcal{R}_j) * \rho_\varepsilon \longrightarrow \Psi'(\beta(f))(\mathcal{R}_j)$ as $\varepsilon \to 0$, for $j = 1, 3$.

In order to conclude the argument leading to point 2), we only need to prove that the term in $(\mathcal{R}_2)$ converges, i.e.

$$(71) \qquad \Psi'(\beta(f) * \rho_\varepsilon)Q(f, \beta(f)) * \rho_\varepsilon \xrightarrow[\varepsilon \to 0]{} \Psi'(\beta(f))Q(f, \beta(f)),$$

where the left-hand side is defined by duality, and the right-hand side is defined in renormalized sense. This is a problem for a fixed function f, but on which we lack any good estimate. All we have at our disposal is the entropy dissipation bound (63), and due to the lack of smoothness of f_*, it is not clear at all that it yields any regularity estimate on $\beta(f) * \rho_\varepsilon$ (think that the convolution has to be in all variables t, x, v for the convergence of the transport term to be ensured). At least an L^2 estimate in the x-variable seems necessary to apply variants of the usual lemmas on commutators between convolution-regularization and differential operators, wich are used in [24] for instance.

Compare this situation with that of [22]. If one wants to use the above line of reasoning to prove vanishing of the defect measure in [22],

then one only needs to show

$$\Psi'(\beta(f) * \rho_\varepsilon)[\Delta\beta(f)] * \rho_\varepsilon \xrightarrow[\varepsilon \to 0]{} \Psi'(\beta(f))\Delta\beta(f)$$
$$= \Delta[\Psi \circ \beta(f)] - \Psi'' \circ \beta(f)|\nabla\beta(f)|^2,$$

where the last term is defined as $-|\nabla(\gamma \circ \beta(f))|^2$, with γ chosen in such a way that $(\gamma')^2 = -\Psi''$. Since $\beta(f) \in L^2_{t,x}(H^1_v)$, we see that $\gamma(\beta(f) * \rho_\varepsilon)$ converges towards $\gamma(\beta(f))$ in the same space, and this implies the conclusion of [22].

On the other hand, in the case of the Boltzmann equation without cut-off, this strategy seems doomed, at least because of the bad dependence of the linear operator $h \longmapsto Q(f, h)$ on the space variable. Our conviction is that there may yet not be enough a priori estimates at our disposal to prove (71). In order to build stronger solutions to the Cauchy problem, it is certainly necessary to go beyond the elementary a priori estimates of mass, energy and entropy disipation, which are at the basis of the theory of renormalized solutions. Of course, we shall work to understand whether the techniques introduced here can help gain further smoothness.

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