ENTROPY DISSIPATION AND LONG-RANGE INTERACTIONS

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Abstract. We study Boltzmann's collision operator for long-range interactions, i.e. without Grad's angular cut-off assumption. We establish a functional inequality showing that the entropy dissipation controls smoothness of the distribution function, in a precise sense. Our estimate is optimal, and gives a unified treatment of both the linear and the nonlinear cases. We also give simple and self-contained proofs of several useful results that were scattered in previous works. As an application, we obtain several helpful estimates for the Cauchy problem, and for the Landau approximation in plasma physics.

Contents

1. Introduction

We study in this work the Boltzmann collision operator, acting on $L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$ by

$$Q(g,f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} B \;\; (g'_* f' - g_* f)\, d\sigma\, dv_*, \tag{1}$$

where $f' = f(v')$ and so on, and

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma \\[2ex] v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma. \end{cases} \tag{2}$$

Here f and g are nonnegative functions of the velocity variable $v \in \mathbb{R}^N$. Moreover, $B(v - v_*, \sigma)$ is a nonnegative function depending only on $|v - v_*|$ and $(\frac{v-v_*}{|v-v_*|}, \sigma)$, and supported in the set $(v - v_*, \sigma) \geq 0$. We note that this last condition is not a restriction when dealing with the nonlinear operator $Q(f, f)$, since (by symmetrization of B) one can always reduce to this case (indeed, $f'f'_*$ is invariant by the transformation $\sigma \to -\sigma$).

The Boltzmann equation is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \tag{3}$$

where the unknown depends on $t \geq 0$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}^N$. If the initial data does not depend on the spatial variable x, then the same holds for all time, and the equation for $f(t, v)$ becomes

$$\frac{\partial f}{\partial t} = Q(f, f). \tag{4}$$

We shall use the notations

$$k = \frac{v - v_*}{|v - v_*|}, \qquad \cos\theta = k \cdot \sigma \quad \left(0 \leq \theta \leq \frac{\pi}{2}\right).$$

We are mainly interested in the case when B has an *angular singularity*, in the sense that for generic relative velocity z, $\int d\sigma B(z, \sigma) = +\infty$. This is the typical situation : in dimension $N = 3$, if the molecular interaction is modelled by an inverse power law (the intensity of the intermolecular force is proportional to r^{-s}, r being the distance between particles), then

$$B(|v - v_*|, \cos\theta) \quad = \quad |v - v_*|^\gamma b(\cos\theta), \tag{5}$$

where $\gamma = (s - 5)/(s - 1)$, and where $b(\cos\theta)$ has a singularity of the form

$$\sin\theta \, b(\cos\theta) \quad \sim \quad \frac{K}{\theta^{1+\nu}} \qquad \text{as} \quad \theta \to 0, \qquad\qquad \nu = 2/(s - 1). \tag{6}$$

Here and below, K denotes various positive constants.

The limiting case $s = 2$, $\nu = 2$ corresponds to Coulomb interaction, and has to be modelled by another operator, see the discussion and references in [34].

More generally than (5), we consider in this paper an arbitrary dimension $N \geq 2$ of velocity space, and collision cross-sections that satisfy

$$B(v - v_*, \sigma) \geq \Phi(|v - v_*|) b(k \cdot \sigma), \tag{7}$$

where the kinetic cross-section $\Phi(|z|) : \mathbb{R}^N \to \mathbb{R}^+$ is continuous and strictly positive for $z \neq 0$, and

$$\sin^{N-2}\theta \, b(\cos\theta) \quad \sim \quad \frac{K}{\theta^{1+\nu}} \qquad \text{as} \quad \theta \to 0, \qquad\qquad \nu > 0. \tag{8}$$

Moreover, we shall see how our main results can be transformed in the case of the still more general condition

$$\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \, b(\cos\theta) \, d\theta = +\infty. \tag{9}$$

On the contrary, the great majority of works on the Boltzmann equation simply remove the angular singularity at $\theta = 0$ and assume that the cross-section is integrable with respect to the angular variable (Grad's angular cut-off assumption). This greatly simplifies the mathematical analysis, but also changes the qualitative behavior of the solutions. In particular, grazing collisions do have *smoothing* effects on the solutions of the Boltzmann equation, as was first observed by Desvillettes in a simplified situation [14], while this is definitely false when the collision cross-section is integrable (the solution of the Boltzmann equation can at best be as regular as the initial data, see eg. [36]).

There is by now a large number of scattered results related to our study, and in this paper we shall endeavor to put some structure into this, and bring together some already existing estimates into a unified framework. Let us be more precise, and, first of all, review briefly the existing literature.

The first mathematical analysis of the "Boltzmann equation without cutoff" seems to be the work of Arkeryd [9], which proves existence of weak solutions to the spatially homogeneous Boltzmann equation (4) in the case $\nu < 1$. Later, Goudon [23] and Villani [35] independently treated the case $\nu < 2$. The minimal condition necessary for a mathematical treatment of the Boltzmann collision operator with a cross-section given by (7) seems to be the requirement

$$\int_{S^{N-1}} b(k \cdot \sigma)(1 - k \cdot \sigma)\, d\sigma < +\infty. \tag{10}$$

In fact, under physically relevant assumptions on Φ, this condition is sufficient to develop a mathematical theory of weak solutions [35, 8]. On the contrary, if it is not satisfied, then the operator (1) simply does not make sense, see [34, Part I, Appendix 1]. Remarkably, the quantity (10) is precisely the main physically meaningful quantity associated to the angular cross-section $b(k \cdot \sigma)$, namely the cross-section for momentum transfer.

The qualitative behavior of solutions of the (spatially homogeneous) Boltzmann equation without cut-off has recently been the object of numerous studies. The main idea is that the Boltzmann operator (1) behaves like a singular integral operator, so that solutions of (4) should become (infinitely) smooth for positive times. Proofs in several particular regimes are due to Desvillettes [14, 15, 16], Proutière [30], Alexandre [6], and A.Pulvirenti [31]. All those proofs are more or less based on Fourier analysis; the analysis of Alexandre also involves pseudo-differential operators.

It is natural to expect that such regularization effects have a "functional version" at the level of the main physically meaningful quantity, namely the entropy dissipation

$$\begin{aligned} D(f) &= -\int_{\mathbb{R}^N} Q(f,f) \log f \, dv \\ &= \frac{1}{4} \int_{\mathbb{R}^{2N}} \int_{S^{N-1}} B(v - v_*, \sigma)(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} \, d\sigma \, dv \, dv_*. \end{aligned} \tag{11}$$

By this we mean that the entropy dissipation, as one of the main physical objects related to the Boltzmann equation (and retaining all of the characteristics of the interaction), is expected to control (some) smoothness of the distribution function in the case of singular cross-sections – and by the a priori estimate $D(f) \in L^1([0,T] \times \mathbb{R}^N_x)$, this would in fact prove a regularizing effect for solutions of the equation.

But the complexity of the functional (11) has obscured this fact for a long time. After the formal study of Alexandre [3], suggesting that smoothness estimates could indeed be deduced from the entropy dissipation, Lions [27] proved a functional inequality of the form

$$\|\sqrt{f}\|^2_{\dot{H}^s(|v|<R)} \leq C_R \|f\|^{\theta}_{L^1} \left(\|f\|_{L^1} + D(f)^{1/2} \right)^{1-\theta} \tag{12}$$

for $s < (\nu/2)(1/(1+\nu/(N-1)))$. His proof was based on the so-called "Q^+ smoothing property", see [11, 28, 24, 36, 37]. Shortly after, Villani [33] obtained the optimal Sobolev exponent $\nu/2$ under the assumption that f is locally bounded below. The method in [33] is completely different, and relies in particular on the Carleman representation, also used independently by Alexandre. Later, it was noticed that previous work of Desvillettes [16] could help relaxing the assumption of lower bound. In the meantime, Alexandre [5] had proven another estimate, with a worse exponent, but independent of the lower bound, and in a form adapted to the study of the so-called asymptotics of grazing collisions (see section 7). All these results are contained in, and greatly simplified by the present work.

Our main result here states that under suitable conditions,

$$\begin{aligned} D(g,f) &\equiv -\int_{\mathbb{R}^N} Q(g,f)\,\log f dv \\ &\geq c_1\|\sqrt{f}\|^2_{H^{\nu/2}} - c_2\|f\|_{L^1_2}\|g\|_{L^1_2}, \end{aligned}$$

for some constants c_1 and c_2 which may depend only on the mass, entropy and energy of f. Note that, by the standard symmetrization argument, $D(f) = D(f,f)$. In particular, this means that we are able to obtain the optimal exponent $s = \nu/2$ without further restrictions (such as lower bounds on f or g) than those that are physically natural.

The full statement of the result is given in Section 2 together with the main steps of the proof. These main steps are a) identification of dominant terms, b) reduction to the so-called Maxwellian case, where the cross-section does not depend on the relative velocity but only on the deviation angle (i.e $\Phi \equiv 1$), and c) the use of Fourier analysis, which completes the proof.

Two of the estimates needed are of independent interest, and could prove useful in many other instances. First, the Cancellation lemma, that can be found in section 3. A more general version of this lemma can be found in [8], but for the sake of completeness we included the proof here, in the form that we need. Secondly, the estimates of the Fourier transform from sections 5, 6, based on a new Plancherel-like formula, are also of interest by themselves.

Then, section 7 describes a number of applications of the main theorem: compactness results for the spatially inhomogeneous Boltzmann equation without cutoff, existence of weak solutions in the spatially homogeneous case, singular limits of equations with cutoff, and the asymptotics of grazing collisions.

On the whole, this paper, together with the strongly related works in preparation [8] (study of equation (3)) and [18] (study of equation (4)) contains and improves by far on the results in [3, 4, 5, 14, 15, 16, 27, 30, 33]. The only noticeable feature that we do not revisit is the pseudo-differential operator point of view developed by Alexandre [1, 2, 4, 5, 6].

The "moral" that should be retained from our estimates is very simple : for a given distribution function $g \in L^1$, *the Boltzmann operator* (1) *behaves like the singular operator* $-(-\Delta)^{\nu/2}$. We wish to make it clear that this is a *linear* effect, in the sense that the regularization properties of the Boltzmann operator are those of the linear operator $f \mapsto$

$Q(g,f)$, and not a particular feature of the nonlinear operator $f \mapsto Q(f,f)$. We note that in the limit case $\nu = 2$, the Boltzmann operator has to be replaced by the Landau operator, which is precisely of diffusion nature.

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2. The main result

Let us introduce the generalized ("linear") entropy dissipation functional

$$D(g,f) = -\int_{\mathbb{R}^N} Q(g,f)\, \log f. \tag{13}$$

Following [8], for a given cross-section B we introduce the quantities

$$\Lambda(|z|) = \int_{S^{N-1}} B(z,\sigma)(1 - k\cdot\sigma)\, d\sigma \tag{14}$$

and

$$\Lambda'(|z|) = \int_{S^{N-1}} B'(z,\sigma)(1 - k\cdot\sigma)\, d\sigma, \tag{15}$$

where

$$B'(z,\sigma) = \sup_{1<\lambda\le\sqrt{2}} \frac{|B(\lambda z,\sigma) - B(z,\sigma)|}{(\lambda - 1)|z|}.$$

The quantity $\Lambda'(z)$ measures in a very mild sense the smoothness of the cross-section with respect to the velocity variable. It is shown in [8] that suitable assumptions on Λ and Λ' are enough to give a sense to a *renormalized formulation of the spatially inhomogeneous Boltzmann equation.*

We also introduce the standard notations

$$\|f\|_{L^1_\alpha} = \int_{\mathbb{R}^N} f(v)(1+|v|)^\alpha\, dv, \qquad \|f\|_{L\log L} = \int_{\mathbb{R}^N} f \log(1+f)\, dv\,.$$

Theorem 1. *Assume that*

$$B(v - v_*, \sigma) \ge \Phi(|v - v_*|)b(k\cdot\sigma), \tag{16}$$

where Φ is continuous, $\Phi(|z|) > 0$ if $z \neq 0$. Assume also that b satisfies the singularity assumption (8)*, and that*

$$\Lambda(|z|) + |z|\Lambda'(|z|) \le C_0(1+|z|)^2.$$

Then, for all $R > 0$ there exists a constant $C_{g,R}$, depending only on b, $\|g\|_{L^1_1}$, $\|g\|_{L\log L}$, R, and on Φ, such that

$$\|\sqrt{f}\|^2_{H^{\nu/2}(|v|<R)} \le C_{g,R}\Big[D(g,f) + \|g\|_{L^1_2}\|f\|_{L^1_2}\Big]. \tag{17}$$

Remark. The precise way in which $C_{g,R}$ depends on g and on R is computable, and can be seen in the proof.

Corollary 1.1. *Assume that $B(z,\sigma) \geq B_0(z,\sigma)$, where $B_0(z,\sigma)$ satisfies the same assumptions as B in Theorem 1. Then,*

$$\|\sqrt{f}\|^2_{H^{\nu/2}(|v|<R)} \leq C_{f,R}\Big[D(f) + \|f\|^2_{L^1_2}\Big]. \tag{18}$$

Remark. Thanks to corollary 1.1, the usual soft potentials without angular cutoff fall in the scope of our study.

Proof of Theorem 1. First we rewrite $D(g,f)$ using the standard pre-postcollisional change of variables $(v, v_*, \sigma) \to (v', v'_*, k)$, which is unitary :

$$\begin{aligned} D(g,f) &= -\int_{\mathbb{R}^{2N}\times S^{N-1}} B(g'_* f' - g_* f)\log f \, dv\, dv_*\, d\sigma \\ &= \int_{\mathbb{R}^{2N}\times S^{N-1}} B\, g_* f \log\frac{f}{f'}\, dv\, dv_*\, d\sigma \\ &= \int_{\mathbb{R}^{2N}\times S^{N-1}} B\, g_*\left(f\log\frac{f}{f'} - f + f'\right) dv\, dv_*\, d\sigma + \int_{\mathbb{R}^{2N}\times S^{N-1}} B\, g_*(f-f')\, dv\, dv_*\, d\sigma\,. \end{aligned}$$

This decomposition splits $D(g,f)$ into two parts, the first of which is signed and retains all the smoothness control. As for the second, it involves strong cancellations due to the presence of the term $f - f'$.

Under our assumptions on the cross-section, the *cancellation lemma* (lemma 1) and its corollary (corollary 1.2), stated and proven in section 3, give a bound for the second term on the right,

$$\int B\, g_*(f'-f)\, dv\, dv_*\, d\sigma \leq C_1\, \|g\|_{L^1_2}\|f\|_{L^1_2}.$$

For the first term we use the inequality

$$x\log\frac{x}{y} - x + y \geq (\sqrt{x}-\sqrt{y})^2\,,$$

which can be proven easily using the fact that it is homogeneous of degree one. Hence

$$D(g,f) + C_1\|g\|_{L^1_2}\|f\|_{L^1_2} \;\geq\; \int B\, g_*\big(\sqrt{f'}-\sqrt{f}\big)^2\, dv\, dv_*\, d\sigma \tag{19}$$

$$\geq\; \int \Phi(|v-v_*|)b(k\cdot\sigma)g_*(\sqrt{f'}-\sqrt{f})^2\, dv\, dv_*\, d\sigma. \tag{20}$$

The aim is now to estimate the expression on the right from below by a Sobolev norm. The calculation is carried out using the Fourier transform, hence an expression involving quadratic terms rather than logarithms is much more adapted.

However, the use of the Fourier transform with the collision operator for the Boltzmann equation is quite complicated unless the kernel B is independent of $|v-v_*|$ (see [30], or the more general computation in the Appendix). Consequently, the next step consists in reducing to this case. The calculation is less technical in the case where $\inf_{|z|<R}\Phi(|z|) > 0$, and hence this will first be considered.

From now on, we let

$$F(v) \quad = \quad \sqrt{f(v)}$$

and the notation F', F_*, etc. is used as for f and g. Moreover, for a given $R > 0$, let $\chi_R(v)$ denote the characteristic function of the ball $B_R = \{|v| < R\}$ (or actually a smooth function such that $0 \le \chi_R \le 1$, $\chi_R \equiv 1$ on B_R, and supp $(\chi_R) \subset B_{R+1}$). As for other functions, $'$ and $*$ are used to indicate where to evaluate the function. The *truncation lemma* (that is, lemma 2) from Section 4 shows that

$$\int B\, g_* \big(\sqrt{f'} - \sqrt{f}\big)^2 \,dv\,dv_*\,d\sigma + C_2 \|f\|_{L^1_2}\|g\|_{L^1_2}$$

$$\geq \min_{|z|\le 2\sqrt{2}R}(1,\Phi(|z|)) \int_{\mathbb{R}^{2N}\times S^{N-1}} b(k\cdot\sigma) g_* \chi_{B_R *} (F'\chi_{B_R}{}' - F\chi_{B_R})^2 \,dv dv_* d\sigma\,. \tag{21}$$

Here B can be estimated from below by b times a constant, because the support of the integrand on the right hand side is contained in $\{|v - v_*| < 2\sqrt{2}R\}$. It it exactly at this point that the R-dependence enters into the estimate (17).

In Proposition 2 (section 5), we establish a Fourier representation of the integral in (21). As a consequence of this representation, we obtain in Corollary 2.1 the inequality

$$\int_{\mathbb{R}^{2N}\times S^{N-1}} b(k\cdot\sigma) g_* \chi_{B*} (F'\chi_{B_R}{}' - F\chi_{B_R})^2 \,dv\,dv_*\,d\sigma$$

$$\geq \int_{\mathbb{R}^N} d\xi\, |\widehat{F\chi_{B_R}}(\xi)|^2 \left\{ \int_{S^{N-1}} d\sigma\, b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \left(\widehat{g\chi_{B_R}}(0) - |\widehat{g\chi_{B_R}}(\xi^-)|\right) \right\}$$

where $\xi^- = (\xi - |\xi|\sigma)/2$.

By Proposition 3 in Section 6 there is a positive constant c_g, which depends only on $\|g\|_{L^1}$, $\|g\|_{L\log L}$, N and b, such that

$$\int_{S^{N-1}} d\sigma\, b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \left(\hat{g}(0) - |\hat{g}(\xi^-)|\right) \geq c_g\, |\xi|^\nu. \tag{22}$$

Therefore

$$\|F\chi_{B_R}\|^2_{H^{\nu/2}} \le 2\big(c_{g\chi_{B_R}} \min_{|z|\le 2\sqrt{2}R}(1,\Phi(|z|))\big)^{-1} \Big(D(g,f) + (C_1 + C_2)\|g\|_{L^1_2}\|f\|_{L^1_2}\Big)\,, \tag{23}$$

which is exactly the statement of the theorem, in the case when B does not vanish near $v = v_*$.

Now, we turn to the more general case in which the kinetic cross-section $\Phi(|z|)$ is allowed to vanish for $|z| = 0$. We wish to prove that the same result holds in this case. In order to do so, we need to truncate small relative velocities. For this we go back to equation (19), and to the truncation lemma. Let v_j be an arbitrary point in $\{|v| \le R\}$, and let r_0 be a small positive number. Then define

$$\begin{aligned} A_j &= \{v \quad | \quad |v - v_j| < r_0/4\} \\ B_j &= \{v \quad | \quad |v - v_j| > r_0, |v| \le R\} \end{aligned}$$

and let χ_{Bj} and χ_{Aj} be the corresponding characteristic functions, mollified in such a way that the supports of χ_{Bj} and χ_{Aj} are separated by a distance $r_0/2$. Then (21) becomes

$$\left[\min_{r_0/2\le|z|\le 2\sqrt{2}R}\big(1,\Phi(|z|)\big)\right]\int_{\mathbb{R}^{2N}\times S^{N-1}}\tilde{b}(k\cdot\sigma)(g\chi_{Bj})_*(F'\chi_{B_R}{}'-F\chi_{B_R})^2\,dv\,dv_*\,d\sigma,$$

while 23 transforms into

$$\|F\chi_{Aj}\|^2_{H^{\nu/2}}\le 2\big(c_{g\chi_{Bj}}\min_{r_0/2\le|z|\le 2\sqrt{2}R}(1,\Phi(|z|))\big)^{-1}\Big(D(g,f)+(C_1+C_2)\|g\|_{L^1_2}\|f\|_{L^1_2}\Big). \tag{24}$$

By choosing r_0 properly, the right-hand side can be made independent of j. The point is that (22) only depends on the mass, entropy and first moment of g, and on b. For any function g with bounded entropy, r_0 can be chosen so that for any $|v_0|<R$,

$$\int_{|v-v_0|<r_0} g(v)\,dv \quad < \quad \frac{1}{2}\int_{|v|<R} g(v)\,dv.$$

Hence, $c_{g,\chi_{Bj}}$ depends on the same quantities as c_g above. The choice of r_0 also changes C_2, which is of the order r_0^{-2}.

All that remains to do is take a set of points $v_j\in\{|v|\le R\}$, $1\le j\le J$, such that

$$B_R \ \subset \textstyle\bigcup_{j=1}^J A_j\,.$$

Then,

$$\begin{aligned}\|F\|^2_{H^{\nu/2}(|v|<R)} &\le J\sum_{j=1}^J\|F\chi_{Aj}\|^2_{H^{\nu/2}}\\ &\le J^2\big(c_g\min_{r_0/2\le|z|\le 2\sqrt{2}R}(1,\Phi(|z|))\big)^{-1}\Big(D(g,f)+(C_1+\tilde{C}_2 r_0^{-2})\|g\|_{L^1_2}\|f\|_{L^1_2}\Big),\end{aligned}$$

which concludes the proof. We have written the last expression rather carefully to show that, though clearly computable, the constants that we have obtained are really much worse in the case when B vanishes as $|v-v_*|\to 0$. A rough estimate shows that it is enough to take $r_0\sim\exp(-4\|g\|_{L\log L}/N)$, and then $J\sim R^N\exp(4\|g\|_{L\log L})$.

□

Remarks.

(1) Note that Theorem 1 yields the right exponent under very general assumptions on the cross section. It also improves considerably on [33] as regards the dependence on the constants with respect to R.

(2) As mentioned above, estimate (17) is completely asymmetric in f and g : smoothness of g does not play any role, while all the differential part bears on f.

(3) In the degenerate case $g=\delta_0$ (the Dirac mass at 0 velocity), the operator $f\mapsto Q(g,f)$ is the adjoint of the linear operator

$$T: f\longmapsto\int_{S^{N-1}}B(v,\sigma)\left[\varphi\left(\frac{v+|v|\sigma}{2}\right)-\varphi(v)\right]d\sigma.$$

More generally, it is the adjoint of the linear operator

$$\int_{\mathbb{R}^N} dv_*\, g_* \left(\tau_{v_*} \circ T \circ \tau_{-v_*}\right), \tag{25}$$

where $\tau_h f(v) = f(v-h)$.

When B is locally integrable and sufficiently smooth, the structure of the operator T was studied by Lions [24] and Wennberg [36, 37]. It is likely that if $B(z,\sigma) = \Phi(|z|)b(k\cdot\sigma)$, where Φ is smooth and non-vanishing, and b satisfies the singularity assumption (8), then T is a pseudo-differential operator of order exactly $\nu/2$. We *avoid* the apparently delicate task of a direct study of T, by showing that any *average* of T, in the sense of (25), is indeed an operator of order $\nu/2$ as soon as the measure g is not purely singular. We do not see (at the moment) any real physical motivation for studying the operator T in itself, but this question is certainly of noticeable mathematical interest.

(4) In the limit case $\nu = 0$, the Sobolev square norm $\|\sqrt{f}\|^2_{H^{\nu/2}}$ can be replaced by

$$\int_{\mathbb{R}^N} \sqrt{f}\,\log(1-\Delta)\sqrt{f}.$$

In fact, as soon as

$$Z(\varepsilon) \equiv \int_\varepsilon^{\frac{\pi}{2}} d\theta\, \sin^{N-2}\theta\, b(\cos\theta) \xrightarrow{\varepsilon\to 0} +\infty, \tag{26}$$

which is the mere definition of the fact that the angular cross-section b is singular, then one gains (local) control of $\sqrt{f}$ in

$$X = \left\{ F \in L^2(\mathbb{R}^N);\quad \int_{|\xi|\geq 1} |\hat{F}(\xi)|^2 Z\left(\frac{1}{|\xi|}\right) d\xi < \infty \right\},$$

so that strong L^2 compactness properties on $\sqrt{f}$ are easily obtained.

3. Cancellation lemma

The final aim of this section is to estimate integrals of the form

$$\int B\, g_*(f'-f)\, dv\, dv_*\, d\sigma.$$

Convergence of such an integral is not obvious, because of the singularity in B for grazing collisions. But near the singularity, one has $v' \simeq v$, and hence one could hope that $f'-f$ vanish in such a way to cancel the singularity. This would certainly be true if f were a smooth function – though only for singularities of order less than 1 –, but not if f is only assumed to lie in L^1. However, due to the particular structure of the integral, it is true on the average, in a sense that is made precise in the Cancellation lemma below.

A variant of this lemma was first proven in a form very close to the one presented here by Villani [33]. Such a lemma is also implicit in Desvillettes, Golse [17], and other variants are proven independently, and exploited in various mathematical situations, by Alexandre [1]. An optimized version is given in [8], by a strategy following that of [33]. In order to make this paper self-contained, we reproduce here the proof in [8], in the form that is useful for us. Actually, in [8], a slightly more precise result is presented, allowing B to contain a singularity of order $|v-v_*|^{-N}$.

For the sake of presentation, in the following statement, we abuse notations by writing $B(v-v_*,\sigma) = B(|v-v_*|,\cos\theta)$, where $\cos\theta = (\frac{v-v_*}{|v-v_*|},\sigma)$ as usual.

Lemma 1 **(Cancellation Lemma).** *For a.e.* $v_* \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N\times S^{N-1}} B(v-v_*,\sigma)(f'-f)\,dv\,d\sigma = (f*S)(v_*), \tag{27}$$

where

$$S(z) = |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \Big[\frac{1}{\cos^N(\theta/2)} B\Big(\frac{|z|}{\cos(\theta/2)},\cos\theta\Big) - B(|z|,\cos\theta)\Big]\,d\theta\,. \tag{28}$$

Moreover, $|S(z)| \le C_N \Big(\Lambda(|z|) + |z|\Lambda'(|z|)\Big)$, *with* Λ *and* Λ' *defined as in (14) and* (15).

Proof. We do the calculation as if B were integrable, and apply a limiting procedure to conclude in the general case (in fact, the right-hand side of (27) should be taken as a definition of the left-hand side). For each σ, and with v_* still fixed, we perform the change of variables $v \to v'$. See figure 1 for an explanation of the notation. Recall that

$$v' \;=\; \frac{v_*+v}{2} + \frac{|v-v_*|}{2}\sigma \;=\; v_* + \frac{|v-v_*|}{2}(k+|k|\sigma).$$

This change of variables is well-defined on the set $(\cos\theta > 0)$, and it follows either by a direct calculation, or by using the cylindrical symmetry of this transformation, that its Jacobian determinant is

$$\left|\frac{dv'}{dv}\right| = \left|\frac{1}{2}I + \frac{1}{2}k\otimes\sigma\right| = \frac{1}{2^N}(1+k\cdot\sigma) = \frac{(k'\cdot\sigma)^2}{2^{N-1}}\,,$$

where $k=(v-v_*)/|v-v_*|$, and $k'=(v'-v_*)/|v'-v_*|$. Then

$$k'\cdot\sigma = \cos\frac{\theta}{2} \ge \frac{1}{\sqrt{2}}.$$

The inverse transformation, $v' \to \psi_\sigma(v') = v$ is then defined accordingly. (Geometrically : draw the hyperplane which bisects the segment $[v_*,v']$, let w be the intersection of this hyperplane with the line issued from v', directed by σ, then v is the velocity which is symmetric of v with respect to w. See fig. 1.)

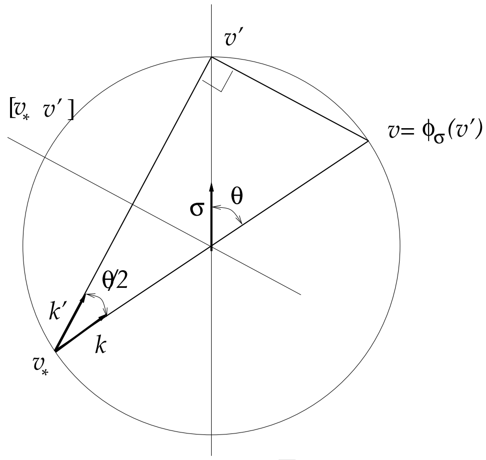

Figure 1

We note that $|v_* - \psi_\sigma(v')| = |v' - v_*|/(k' \cdot \sigma)$, or (which is the same)

$$|v_* - \psi_\sigma(v)| = \frac{|v - v_*|}{k \cdot \sigma}.$$

Applying this change of variable to the part $\int Bf'$ in the left-hand side of (27), and then changing the name v' for v, we find

$$\int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \sigma) f(v')\, dv\, d\sigma = \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_*, \sigma) f(v') \left|\frac{dv}{dv'}\right| dv'\, d\sigma$$

$$= \int_{k \cdot \sigma \geq 1/\sqrt{2}} B(\psi_\sigma(v) - v_*, \sigma) f(v') \frac{2^{N-1}}{(k' \cdot \sigma)^2}\, dv'\, d\sigma.$$

Thus (27) holds with

$$S(v - v_*) = \int_{k \cdot \sigma \geq \frac{1}{\sqrt{2}}} \frac{2^{N-1}}{(k \cdot \sigma)^2} B\big(|v_* - \psi_\sigma(v)|, 2(k \cdot \sigma)^2 - 1\big) d\sigma$$
$$- \int_{k \cdot \sigma \geq 0} B(|v - v_*|, k \cdot \sigma) d\sigma.$$

The first part of this is

$$\int_{k \cdot \sigma \geq \frac{1}{\sqrt{2}}} \frac{2^{N-1}}{(k \cdot \sigma)^2} B\big(\frac{|v - v_*|}{k \cdot \sigma}, 2(k \cdot \sigma)^2 - 1\big) d\sigma$$
$$= |S^{N-2}| \int_0^{\frac{\pi}{4}} \sin^{N-2}\theta \, \frac{2^{N-1}}{\cos^2\theta} B\left(\frac{|v - v_*|}{\cos\theta}, \cos(2\theta)\right) d\theta$$
$$= |S^{N-2}| \int_0^{\frac{\pi}{2}} \frac{\sin^{N-2}\theta}{\cos^N(\theta/2)} B\left(\frac{|v - v_*|}{\cos(\theta/2)}, \cos\theta\right)\, d\theta,$$

and finally

$$S(v-v_*) = |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \Big[\frac{1}{\cos^N(\theta/2)} B\Big(\frac{|v-v_*|}{\cos(\theta/2)}, \cos\theta\Big) - B(|v-v_*|,\cos\theta)\Big]\, d\theta\,. \tag{29}$$

Here the two terms within the brackets are singular near $\cos\theta = 1$, but then they essentially only differ in the first variable. This fact is now exploited to find that (29) is bounded by

$$|S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \frac{1}{\cos^N(\theta/2)} \left| B\left(\frac{|v-v_*|}{\cos(\theta/2)}, \cos\theta\right) - B(|v-v_*|,\cos\theta)\right| d\theta$$
$$+ |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \left[\frac{1}{\cos^N(\theta/2)} - 1\right] B(|v-v_*|,\cos\theta)\, d\theta\,.$$

$$\le |S^{N-2}| 2^{\frac{N}{2}} \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \big(1-\cos(\theta/2)\big)|v-v_*| B'(|v-v_*|,\cos\theta)\, d\theta$$
$$+ |S^{N-2}| 2^{N/2} \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \big(1-\cos^N(\theta/2)\big) B(|v-v_*|,\cos\theta)\, d\theta\,.$$

Since $1-\cos^N(\theta/2) \le N\big(1-\cos(\theta/2)\big) = 2N\sin^2(\theta/4)$, there is a cancellation of order 2, and we can conclude the proof by recalling the definitions of Λ, Λ'. □

Remark. It is interesting to note that for cross sections of the form $B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma \cos\theta$, the function $S(|v-v_*|)$ has a definite sign. In this case (29) is

$$S(v-v_*) = |S^{N-2}||v-v_*|^\gamma \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \Big[|\cos(\theta/2)|^{-(\gamma+N)} - 1\Big] b(\cos\theta)\, d\theta$$

which is nonnegative if $\gamma > -N$. "Coulomb" potentials with $\gamma = -N$ require a more careful analysis.

The cancellation lemma directly gives an estimate of the expression in the beginning of the section. We state this slightly more generally here than in Section 2, where it is used.

Corollary 1.2. *Assume that Λ and Λ' are defined by (14) and (15) respectively, and that $\Lambda(|v-v_*|) + \Lambda'(|v-v_*|) \le C(|v-v_*|^\gamma + |v-v_*|^2)$.*

(1) *If $0 \le \gamma \le 2$, then*

$$\left|\int B\, g_*(f'-f)\, dv\, dv_*\, d\sigma\right| \;\le\; C\|g\|_{L^1_2}\|f\|_{L^1_2}.$$

(2) *If $-N < \gamma < 0$, then*

$$\left|\int B\, g_*(f'-f)\, dv\, dv_*\, d\sigma\right| \;\le\; C\|g\|_{L^{p_1}_2}\|f\|_{L^{p_2}_2}, \tag{30}$$

where $p_1^{-1} + p_2^{-1} = 2 + \gamma/N$

Proof. This is a direct consequence of the cancellation lemma. A bilinear form of the Hardy-Littlewood-Sobolev inequality (see [32]) implies (30). □

4. Truncation

This section deals with with the expression

$$\int B\, g_*\big(\sqrt{f'}-\sqrt{f}\big)^2\, dv\, dv_*\, d\sigma\,. \tag{31}$$

As before we assume that $B(|v-v_*|,\cos\theta) \geq \Phi(|v-v_*|)b(\cos\theta)$. The aim is to replace B by b, which simplifies the analysis of (31) considerably.

Recall that F and F' denote respectively $\sqrt{f}$ and $\sqrt{f'}$, and that χ_{B_R} denotes the (mollified) characteristic function of the ball $\{|v|\leq R\}$, whereas χ_{A_j} and χ_{B_j} denote the (mollified) characteristic functions corresponding to the sets A_j and B_j. These sets are defined below.

Lemma 2. (1) *Assume that $\Phi(|v-v_*|)$ is continuous and bounded from below for all $|v-v_*|\leq R$. Then*

(32)
$$\int B\, g_*\big(\sqrt{f'}-\sqrt{f}\big)^2\, dv\, dv_*\, d\sigma + C_2\|f\|_{L^1_2}\|g\|_{L^1_2}$$
$$\geq \min_{|z|\leq 2\sqrt{2}R}(1,\Phi(|z|)) \int_{\mathbb{R}^{2N}\times S^{N-1}} b(k\cdot\sigma)(g\chi_B)_*((F\chi_{B_R})' - F\chi_{B_R})^2\, dv\, dv_*\, d\sigma$$

where the constant C_2 depends only on N and B.

(2) *Assume that Φ is as above, except that it may vanish as $|z|\to 0$. Let $r_0>0$ be a (small) positive constant, and let $A_j=\{v\in\mathbb{R}^N;\ |v-v_j|\leq r_0/4\}$, where $|v_j|<R$. Moreover, let $B_j=\{v\in\mathbb{R}^N;\ |v|\leq R,\ \ |v-v_j|\geq r_0\}$. Then,*

(33)
$$\int B\, g_*\big(\sqrt{f'}-\sqrt{f}\big)^2\, dv\, dv_*\, d\sigma + C_2 r_0^{-2}\|f\|_{L^1_2}\|g\|_{L^1_2}$$
$$\geq \min_{\frac{r_0}{2}\leq|z|\leq 2\sqrt{2}R}(1,\Phi(|z|)) \int_{\mathbb{R}^{2N}\times S^{N-1}} b(k\cdot\sigma) g_*\chi_{B_j}(F'\chi_{A_j}{}' - F\chi_{A_j})^2\, dv dv_* d\sigma$$

Proof. We begin by estimating the effect of replacing f by $f\chi_A$ and g by $g\chi_B$, where χ_A denotes a (smoothed) characteristic function of the set $A\subset\mathbb{R}^N$. Clearly

$$g_*(F'-F)^2 \;\;\geq\;\; g_*\chi_{B*}(F'-F)^2\chi_A^2$$

The inequality

$$\begin{aligned}(F'\chi_A' - F\chi_A)^2 &= (F'(\chi_A'-\chi_A) + (F'-F)\chi_A)^2\\ &\leq 2F'^2(\chi_A'-\chi_A)^2 + 2(F'-F)^2{\chi_A}^2\,,\end{aligned}$$

shows that the integrand of (31) can be estimated in the following way:

$$\begin{aligned}&2\,\Phi(|v-v_*|)b(k\cdot\sigma)g_*(F'-F)^2\\ &\qquad\geq \min(1,\Phi(|v-v_*|))\, b(k\cdot\sigma)\, g_*\chi_{B*}(F'-F)^2\chi_A^2\\ &\qquad\geq \frac{1}{2}\min(1,\Phi(|v-v_*|))\, b(k\cdot\sigma)\, g_*\chi_{B*}(F'\chi_A' - F\chi_A)^2\\ &\qquad\qquad - b(k\cdot\sigma)g_*\chi_{B*}{F'}^2(\chi_A'-\chi_A)^2.\end{aligned} \tag{34}$$

In the last term,

$$(\chi_A' - \chi_A)^2 \quad \le \quad \|\nabla \chi_A\|_{L^\infty}^2 |v - v'|^2 = \|\nabla \chi_A\|_{L^\infty}^2 |v - v_*|^2 \sin^2 \frac{\theta}{2},$$

and in the case where A is a ball of radius $\mathrm{diam}(A)$, it is possible to regularize χ_A in such a way that the support of χ_A is contained in a ball with diameter not larger than $1.01 \times \mathrm{diam}(A)$, and $\|\nabla \chi_A\|_{L^\infty}^2 < C \max(1, (\mathrm{diam}(A)))^{-2}$. Denoting any constant depending on A in this way by C_A,

$$\begin{aligned}
&\int g_* \chi_{B*} b(k \cdot \sigma) f' (\chi_A - \chi_A')^2 dv \; dv_* \, d\sigma \\
&\qquad \le C_A \int g_* |v - v_*|^2 b(\cos\theta) \sin^2 \frac{\theta}{2} f' \, dv \, dv_* \, d\sigma \\
&\qquad \le C_A \int_{|\theta| \le \frac{\pi}{2}} g_* |v' - v_*|^2 b(\cos\theta) \cos^{-4} \frac{\theta}{2} \sin^2 \frac{\theta}{2} f' \, 2^N \, dv' \, dv_* \, d\sigma
\end{aligned} \tag{35}$$

$$\le C_A \|f\|_{L^1_2} \|g\|_{L^1_2}.$$

It remains to study the first term in (34). Note first that

$$|v' - v_*| = \cos \frac{\theta}{2} \, |v - v_*|,$$

and that $|\theta| \le \frac{\pi}{2}$. Therefore,

$$|v' - v_*| \le |v - v_*| \le \sqrt{2} \, |v' - v_*|. \tag{36}$$

For the first estimate of the lemma, it is enough to take the sets $B = A = \{|v| \le R\}$. As a consequence of estimate (36), we see that when $|v_*| \le R$, $|v - v_*|^2 < 8R^2$ as soon as $|v| \le R$ or $|v'| \le R$. Simply integrating (34) and moving the second term to the left gives

$$\begin{aligned}
&2 \int_{\mathbb{R}^{2N} \times S^{N-1}} \min(1, \Phi(|v - v_*|)) b(k \cdot \sigma) g_* (F' - F)^2 \, dv \, dv_* d\sigma + C_A \|f\|_{L^1_2} \|g\|_{L^1_2} \\
&\quad \ge \min_{|z| \le 2\sqrt{2}R} (1, \Phi(|z|)) \int_{\mathbb{R}^{2N} \times S^{N-1}} b(k \cdot \sigma) g_* \chi_{B*} (F' \chi_A' - F \chi_A)^2
\end{aligned} \tag{37}$$

For the second part of the lemma we need to remove a diagonal $|v - v_*| < r_0$, and show that this does not in an essential way change the result. To this end we begin by choosing a (small) constant r_0, and a $v_j \in \chi_{B_R}$. These together determine the sets A_j and B_j. From their definition, it follows that A_j and B_j are separated by a distance of $3r_0/4$, and that if the characteristic functions of A_j and B_j are replaced by suitably mollified functions, the supports of these may still be separated by a distance of $r_0/2$ (see fig. 2)

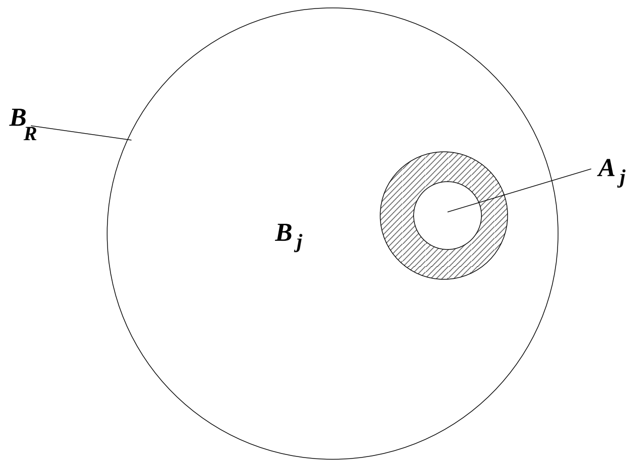

Figure 2

Thanks to estimate (36), we know that

$$\begin{aligned} v_* \in B_j, v \in A_j \quad &\Longrightarrow \quad |v_* - v| > r_0/2, \\ v_* \in B_j, v' \in A_j \quad &\Longrightarrow \quad |v_* - v| > r_0/2. \end{aligned}$$

The inequality (37) now holds with the sets A and B replaced by A_j and B_j respectively, and with $\min_{|z|\le 2\sqrt{2}R}(1, \Phi(|z|))$ replaced by $\min_{r_0/2\le|z|\le 2\sqrt{2}R}(1, \Phi(|z|))$. Note that in this case, C_A depends on the radius r_0.

□

5. Fourier transform

We denote by

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^N} e^{-iv\cdot\xi} f(v)\, dv$$

the Fourier transform of a function $f \in L^1(\mathbb{R}^N)$.

Proposition 2. *The following Plancherel-type identity holds for arbitrary functions $g \in L^1(\mathbb{R}^N)$, $F \in L^2(\mathbb{R}^N)$:*

$$\int_{\mathbb{R}^{2N}} \int_{S^{N-1}} b(k\cdot\sigma) g_* (F' - F)^2 \, dv\, dv_*\, d\sigma$$

$$(38) \qquad = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big) \Big[\hat{g}(0)|\hat{F}(\xi)|^2 + \hat{g}(0)|\hat{F}(\xi^+)|^2 - \hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) - \overline{\hat{g}}(\xi^-)\overline{\hat{F}}(\xi^+)\hat{F}(\xi)\Big]\, d\xi\, d\sigma\,,$$

where

$$(39) \qquad \xi^+ = \frac{\xi + |\xi|\sigma}{2}, \qquad \xi^- = \frac{\xi - |\xi|\sigma}{2}.$$

Proof. Again, we shall do the proof only in the case when b is integrable, and the result will follow by monotonicity.

Expanding the quadratic term in (38) gives three terms,

$$(40) \qquad F'^2 - 2FF' + F^2\,.$$

We begin with the middle term. By the pre-postcollisional change of variables, and Parseval's identity,

$$\begin{aligned}\int b(k\cdot\sigma) g_* F' F \, dv\, dv_*\, d\sigma &= \int Q^+(g,F) F\, dv \\ &= \frac{1}{(2\pi)^N} \int \mathcal{F}\Big[Q^+(g,F)\Big] \overline{\hat{F}} d\xi\,.\end{aligned}$$

Then, we invoke **Bobylev's identity**[1] [10] :

$$\mathcal{F}\Big[Q^+(g,F)\Big] = \int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi^-)\hat{F}(\xi^+)\, d\sigma\,, \tag{41}$$

to deduce that

$$\int b(k\cdot\sigma) g_* F' F\, dv\, dv_*\, d\sigma = \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) d\xi\, d\sigma\,.$$

Of course, this expression is also equal to its own complex conjugate. This shows how to compute the cross-products in (38).

Next, we note that, since $\int_{S^{N-1}} b(k\cdot\sigma)\, d\sigma$ does not depend on the unit vector k,

$$\begin{aligned}\int b(k\cdot\sigma) g_* F^2\, dv\, dv_*\, d\sigma &= \int b(k\cdot\sigma)\, d\sigma \int g_*\, dv_* \int F^2\, dv \\ &= \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(0)|\hat{F}|^2(\xi) d\sigma d\xi,\end{aligned} \tag{42}$$

where we have applied the usual Plancherel identity.

For the term involving F'^2, we first make the change of variables $(v,v_*)\to(v-v_*,v_*)$, and then $v\to v'$ as in Section 3 to obtain

$$\begin{aligned}&\int\int g(v_*) b(\frac{v}{|v|}\cdot\sigma)\Big|\tau_{-v_*}F(\frac{v+|v|\sigma}{2})\Big|^2 dv\, d\sigma\, dv_* \\ &\qquad = \int\int g(v_*) b(\psi(v',\sigma)) \frac{2^{N-1}}{\left(\frac{v'}{|v'|}\cdot\sigma\right)^2} |\tau_{-v*}F(v')|^2\, dv'\, d\sigma\, dv_*\,,\end{aligned}$$

where

$$\psi(v',\sigma) = 2\left(\frac{v'}{|v'|}\cdot\sigma\right)^2 - 1\,,$$

and $\tau_{-v_*}F = F(v_* + \cdot)$. Because $|\mathcal{F}(\tau_h F)| = |\mathcal{F}(F)|$, and using the fact that $\int_{S^{N-1}} b(k\cdot\sigma)\, d\sigma$ does not depend on k, we obtain

$$\frac{1}{(2\pi)^N}\int g(v_*)\Big(\int b(\psi(\xi,\sigma))\frac{2^{N-1}}{\left(\frac{\xi}{|\xi|}\cdot\sigma\right)^2}|\hat{F}(\xi)|^2 d\xi\, d\sigma\Big) dv_*\,.$$

[1]For the sake of completeness, a proof of this identity is included in an appendix. The result there is more general, and includes Maxwellian molecules as a special case.

Finally we note that the inner integral does not depend on v_*, so that, reversing the change of variables from Section 3, we can rewrite the last expression as

$$\frac{1}{(2\pi)^N}\hat{g}(0)\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left|\hat{F}\left(\frac{\xi+|\xi|\sigma}{2}\right)\right|^2 d\xi\,d\sigma.$$

Putting all the pieces together, we conclude the proof of the identity. □

Corollary 2.1. *For all $f \in L^1(\mathbb{R}^N)$, $f \geq 0$, and $g \in L^2(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^{2N}}\int_{S^{N-1}} b(k\cdot\sigma)g_*(F'-F)^2\,dv\,dv_*\,d\sigma$$
$$\geq \frac{1}{2(2\pi)^N}\int_{\mathbb{R}^N}|\hat{F}(\xi)|^2\left\{\int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(0)-|\hat{g}(\xi^-)|\right)d\sigma\right\}d\xi\,.$$

Proof. This is a simple consequence of Young's inequality and of the estimate

$$|\hat{F}(\xi)|^2+|\hat{F}(\xi^+)|^2 \geq |\hat{F}(\xi)|^2.$$

□

6. Decrease in Fourier space

The following proposition (which is a generalization of estimate (3.15) in Desvillettes [16]), is the last piece needed to complete the proof of Theorem 1.

Proposition 3. *Suppose that b satisfies assumption 8. Then, there exists a positive constant C_g, depending only on N, $\|g\|_{L^1_1}$, $\|g\|_{L\log L}$ and b, such that for $|\xi| \geq 1$,*

$$\int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(0)-|\hat{g}(\xi^-)|\right)d\sigma \geq C_g\,|\xi|^\nu. \tag{43}$$

This proposition is itself a consequence of the two lemmas below.

Lemma 3. *There exists a positive constant C'_g, depending only on N, $\|g\|_{L\log L}$, $\|g\|_{L^1_1}$, such that for all $\xi \in \mathbb{R}^N$,*

$$\hat{g}(0)-|\hat{g}(\xi)| \geq C'_g\,(|\xi|^2\wedge 1).$$

Lemma 4. *There exists a constant $K(\nu)$, such that if*

$$\sin^{N-2}\theta\, b(\cos\theta) \;\sim\; \frac{K}{\theta^{1+\nu}} \qquad as \quad \theta\to 0, \qquad\qquad \nu>0$$

then for all $\xi \in \mathbb{R}^N$, $|\xi| \geq 1$,

$$\int_{S^{N-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(|\xi^-|^2\wedge 1)\,d\sigma \;\geq K(\nu)|\xi|^\nu.$$

Proof of lemma 3. Note first that for some $\theta \in \mathbb{R}$,

$$\hat{g}(0)-|\hat{g}(\xi)| = \int_{\mathbb{R}^N} g(v)\,(1-\cos(v\cdot\xi+\theta))\,dv$$
$$= 2\int_{\mathbb{R}^N} g(v)\,\sin^2\left(\frac{v\cdot\xi+\theta}{2}\right)\,dv$$

$$\geq 2\,\sin^2\varepsilon\int_{\{|v|\leq r,\forall p\in\mathbb{Z},|v\cdot\xi+\theta-2\,p\,\pi|\geq 2\,\varepsilon\}} g(v)\,dv$$

$$\geq 2\,\sin^2\varepsilon\Big\{||g||_{L^1(\mathbb{R}^N)}-\frac{||g||_{L^1_1(\mathbb{R}^N)}}{r}-\int_{\{|v|\leq r,\exists p\in\mathbb{Z},|v\cdot\frac{\xi}{|\xi|}+\frac{\theta}{|\xi|}-p\,\frac{2\,\pi}{|\xi|}|\leq 2\,\frac{\varepsilon}{|\xi|}} g(v)\,dv\Big\}$$

$$\geq 2\,\sin^2\varepsilon\Big\{||g||_{L^1(\mathbb{R}^N)}-\frac{||g||_{L^1_1(\mathbb{R}^N)}}{r}-\sup_{|A|\leq\frac{4\,\varepsilon}{|\xi|}\,(2\,r)^{N-1}\,(1+\frac{r\,|\xi|}{\pi})}\int_A g(v)\,dv\Big\}. \tag{44}$$

When $|\xi|\geq 1$, we obtain our lemma with

$$C'_g=2\,\sin^2\varepsilon\Big\{||g||_{L^1(\mathbb{R}^N)}-\frac{||g||_{L^1_1(\mathbb{R}^N)}}{r}-\sup_{|A|\leq 4\,\varepsilon\,(2\,r)^{N-1}+\frac{2\varepsilon}{\pi}(2\,r)^N}\int_A g(v)\,dv\Big\},$$

$\varepsilon>0$ and $r>0$ being chosen in such a way that this quantity is positive.

When $|\xi|\leq 1$, we put $\delta=\frac{\varepsilon}{|\xi|}$ in (44), and set

$$C'_g=2\,\delta^2\inf_{|\xi|\leq 1}\left|\frac{\sin^2(\delta\,|\xi|)}{\delta^2\,|\xi|^2}\right|$$
$$\times\Big\{||g||_{L^1(\mathbb{R}^N)}-\frac{||g||_{L^1_1(\mathbb{R}^N)}}{r}-\sup_{|A|\leq 4\,\delta\,(2\,r)^{N-1}\,(1+\frac{r}{\pi})}\int_A g(v)\,dv\Big\},$$

$\delta>0$ and $r>0$ being chosen in such a way that this quantity is positive.

□

Proof of lemma 4. We first note that

$$|\xi^-|^2=\frac{|\xi|^2}{2}\left(1-\frac{\xi}{|\xi|}\cdot\sigma\right).$$

Passing to N-dimensional spherical coordinates, we find for some $\theta_0>0$,

$$\int_{S^{N-1}} b(\frac{\xi}{|\xi|}\cdot\sigma)(|\xi^-|^2\wedge 1)\,d\sigma=|S^{N-2}|\int_0^{\frac{\pi}{2}}\sin^{N-2}\theta\,b(\cos\theta)\left[\frac{|\xi|^2}{2}(1-\cos\theta)\wedge 1\right]d\theta$$
$$\geq\frac{K}{2}|S^{N-2}|\int_0^{\theta_0}\left(\frac{|\xi|^2\theta^2}{2}\wedge 1\right)\frac{d\theta}{\theta^{1+\nu}}.$$

By the change of variables $\theta\to|\xi|\theta$, this integral is also

$$|\xi|^\nu\int_0^{\theta_0}\left(\frac{\theta^2}{2}\wedge 1\right)\frac{d\theta}{\theta^{1+\nu}},$$

so that when $|\xi|\geq 1$, lemma 4 holds with

$$K(\nu)=\frac{K}{2}|S^{N-2}|\int_0^{\theta_0}(\frac{\theta^2}{2}-1)\,\frac{d\theta}{\theta^{1+\nu}}.$$

□

Remark. In the limit case $\nu = 0$, we also find the lower bound

$$\int_{\sqrt{2}}^{\frac{\pi}{2}|\xi|} \frac{d\theta}{\theta} \sim \log|\xi| \text{ as } |\xi| \to \infty.$$

And more generally,

$$\int_{S^{N-1}} b(k\cdot\sigma)(|\xi^-|^2 \wedge 1)d\sigma \geq \frac{|S^{N-2}|}{4\pi^2}\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta\, b(\cos\theta)(|\xi|^2\theta^2 \wedge 1)\, d\theta$$
$$\geq \frac{|S^{N-2}|}{4\pi^2}\int_{\frac{1}{|\xi|}}^{\frac{\pi}{2}} \sin^{N-2}\theta\, b(\cos\theta)\, d\theta\,.$$

This computation enables to treat arbitrary singularities.

7. Some applications

We now give some applications of Theorem 1 to the study of the Boltzmann equation. Several of them were already known to hold in certain particular regimes, others are new.

Regularity for the spatially homogeneous Boltzmann equation without cut-off

Let $B(v - v_*, \sigma) = |v - v_*|^\gamma b(k\cdot\sigma)$, with $-4 < \gamma \leq 1$, and b satisfying the singularity condition (8) as well as the requirement (10). It is shown in [35] that given an initial datum f_0 satisfying

$$\int_{\mathbb{R}^N} f_0(v)\big(1 + |v|^2 + \log f_0(v)\big)\, dv < +\infty,$$

(and $f_0 \in L^1_{2+\delta}$ for some $\delta > 0$ if $\gamma > 0$), one can construct "weak" solutions (so-called H-solutions for $\gamma < -2$) in $C(\mathbb{R}^+, \mathcal{D}'(\mathbb{R}^N_v))$, which preserve mass and energy, and satisfy the following entropy dissipation inequality : for any $T > 0$,

$$H(f(T,\cdot)) + \int_0^T D(f(t,\cdot))\, dt \leq H(f_0).$$

where $H(f) = \int_{\mathbb{R}^N} f \log f$.

A straightforward application of Theorem 1 shows that these solutions satisfy the smoothness estimate

$$\sqrt{f} \in L^2([0,T]; H^{\nu/2}_{\text{loc}}(\mathbb{R}^N_v)). \tag{45}$$

"Classical" weak solutions for the spatially homogeneous Boltzmann equation with very soft potentials

Estimate (45) can also be used as an a priori estimate to construct weak solutions, in a classical sense, to the spatially homogeneous Boltzmann equation, in the delicate case when the cross-section presents a strong singularity in the relative velocity variable. In [35], weak solutions in an appropriate sense were constructed by the use of Boltzmann's H-theorem. They were called H-solutions. This notion is useful in the case of so-called very soft potentials, i.e when the kinetic cross-section behaves like $|v - v_*|^\gamma$, $\gamma < -2$.

As explained in [35], the apparent obstruction to the definition of more standard weak solutions was the lack of an a priori estimate of the form

$$f f_* |v - v_*|^{\gamma+2} \in L^1_{\rm loc}(dv\, dv_*\, dt).$$

Our present work provides such an estimate in the case when the kinetic singularity is compensated by a *strong enough angular singularity*, in the following sense : the kinetic singularity is of order at most γ, the angular singularity of order at least $1+\nu$ (with our usual notations), and

$$\gamma + \nu + 2 \geq 0. \tag{46}$$

Indeed, under this assumption we can write, applying the Hardy–Littlewood–Sobolev inequality, interpolation in Lebesgue spaces and (fractional) Sobolev embedding,

$$\begin{aligned} \int f f_* |v - v_*|^{\gamma+2} &\leq C \|f\|_{L^\infty_t(L^1_v)} \|f\|_{L^1_t(L^q_v)} \\ &\leq C \|f_0\|_{L^1} \|\sqrt{f}\|^2_{L^2_t(H^{\nu/2}_v)}, \end{aligned}$$

for some well-chosen q, where everything has to be understood in local sense.

It is worth pointing out that inequality (46) *always* holds for cross-sections coming out from inverse-power forces in dimension 3. We also note that the case which appears the most delicate to treat now, is the one of a cross-section which is singular in the relative velocity variable but not in the angular variable. See [8] for a related problem.

Strong compactifying effects for solutions of the spatially inhomogeneous Boltzmann equation without cut-off

As explained in Lions [27], a smoothness estimate in the v variable like the one in Theorem 1, combined with a so-called renormalized formulation of the spatially inhomogeneous equation (3), is enough to prove that solutions (or approximate solutions) (f_n) of (3) enjoy a property of immediate strong compactification, in the following sense. If the sequence of initial data $(f_0^n)_{n\in\mathbb{N}}$ satisfies only the physically natural bounds

$$\sup_{n\in\mathbb{N}} \int f_0^n(x,v)\big(1 + |x|^2 + |v|^2 + \log f_0^n(x,v)\big)\, dx\, dv < +\infty,$$

(and is therefore weakly compact in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$), then for all time $t > 0$ the sequence $(f^n(t,\cdot,\cdot))$ is *strongly* compact in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$ (i.e., converges a.e., up to extraction).

The strategy runs as follows : first, by the use of a renormalized formulation [19, 20, 26], and velocity-averaging lemmas [22, 21], one proves that suitable quantities of the form $\beta(f^n) *_v \phi_\delta$, where ϕ_δ ($\delta > 0$) is a mollifier in the velocity space only, are strongly compact. Then, by truncation arguments, the smoothness estimate in v applies out of a set of small measure in (t,x), where $\|f^n(t,x,\cdot)\|_{L^1_2}$ or $\|f^n(t,x,\cdot)\|_{L\log L}$ may be infinite, and out of a set where the L^1 norm of $f^n(t,x,\cdot)$ is very small. Out of these particular sets, the velocity smoothness entails that $\beta(f^n) *_v \phi_\delta$ is very close to $\beta(f^n)$, uniformly in n, as δ goes to 0, and this is enough to prove strong compactness of $\beta(f^n)$, which in turn implies pointwise convergence of f^n if β is chosen to be one–to–one.

In [8] a general renormalized formulation of the Boltzmann equation will be introduced, which will enable to carry out this program. We refer to [8] for more details, precise statements and complete proofs of the strong compactification property of the Boltzmann equation without cut-off (which was conjectured in Lions [25, 26]).

Singular limit of the cutoffed Boltzmann equation

In fact, and this is important to prove existence results for the non-cutoff Boltzmann equation, strong compactification as above holds even if one considers a sequence of solutions (f^n) to the Boltzmann equation with respective cross-section B_n, where B_n converges in an appropriate sense towards a singular kernel B as $n \to \infty$. We emphasize that this statement is of interest even in the spatially homogeneous case. Even if for each fixed n there is no smoothing effect, on the whole the sequence (f^n) will be compact.

Precise statements and complete proofs are again performed in [8]. They rely on the following general estimate. If $B_n(v - v_*, \sigma) \geq \Phi_0(|v - v_*|) b_n(k \cdot \sigma)$, where Φ_0 is a fixed continuous function, $\Phi_0(|z|) > 0$ for $|z| \neq 0$, and if D_n is the entropy dissipation functional associated to the cross-section B_n, then holds the estimate (for arbitrary functions $f^n(v)$)

$$\int_{|\xi|\geq 1} |\mathcal{F}[\sqrt{\chi_R f_n}](\xi)|^2 \, Z_n\left(\frac{1}{|\xi|}\right) d\xi \leq C(f^n, R, \Phi_0) \Big[D_n(f^n) + \|f^n\|^2_{L^1_2}, \Big],$$

where

$$Z_n(\varepsilon) = |S^{N-2}| \int_\varepsilon^{\frac{\pi}{2}} \sin^{N-2}\theta \, b_n(\cos\theta) \, d\theta \,.$$

Here χ_R is a cutoff function with support in B_{R+1}, identically equal to 1 on B_R, and $C(f^n, R, \Phi_0)$ is a constant depending on χ_R and Φ_0, and on f^n only through $\|f^n\|_{L^1}$ (which must be nor too small neither too large), $\|f^n\|_{L \log L}$, $\|f^n\|_{L^1_2}$ (which must not be too large).

This estimate is a direct consequence of our proof.

A typical example is when $b_n(\cos\theta) = b(\cos\theta) 1_{\theta \geq n^{-1}}$, and b satisfies the singularity condition (8). In that case,

$$Z_n\left(\frac{1}{|\xi|}\right) \geq C\, (|\xi| \wedge n)^\nu$$

for some positive constant C as soon as $|\xi| \geq 1$. In the limit case $\nu = 0$, the right-hand side can be replaced by $\min(\log|\xi|, \log n)$. And more generally, by $Z(\max(|\xi|^{-1}, n^{-1}))$, if Z is defined by (26).

With still more generality, it is shown in [8] that compactness holds as soon as

$$\int_{S^{N-1}} \underline{\lim}\, b_n(k \cdot \sigma) \, d\sigma = +\infty.$$

The asymptotics of grazing collisions and the Debye cut

The asymptotics of grazing collisions [13, 23, 35] are the mathematical formulation of the approximation of the Boltzmann equation by the *Landau equation*, in the regime when grazing collisions prevail. The motivation for this approximation comes from plasma physics. Indeed, for Coulomb interaction, in dimension $N = 3$, apart from numerical constants,

$$B(v - v_*, \sigma) = |v - v_*|^{-3} \, b(k \cdot \sigma), \tag{47}$$

$$b(\cos\theta) = b^C(\cos\theta) \sim \frac{1}{\sin\theta}\frac{1}{\theta^3} \quad \text{as } \theta \to 0.$$

Therefore the integral

$$\Lambda^C = 2|S^1| \int_0^{\frac{\pi}{2}} \sin\theta\, b^C(\cos\theta)\sin^2\frac{\theta}{2}\,d\theta \sim 16\pi \int \frac{d\theta}{\theta}$$

is logarithmically divergent for small deviation angles.

A general framework for the grazing collisions limit is given in Villani [34]. We say that the sequence of kernels $b_n(\cos\theta)$ concentrates on grazing collisions if

$$\begin{cases} \Lambda_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta\, b_n(\cos\theta)\,(1-\cos\theta)\,d\theta \xrightarrow[n\to\infty]{} \Lambda \in (0,+\infty); \\ \\ \text{for all } \theta_0 > 0, \quad \displaystyle\sup_{\theta\geq\theta_0} b_n(\cos\theta) \xrightarrow[n\to\infty]{} 0. \end{cases} \tag{48}$$

This general formulation covers all existing asymptotics, though they may look quite different at first sight [12, 13].

It can be shown under very general assumptions that solutions of the spatially homogeneous Boltzmann equation with, say, kernel $B_n(v-v_*,\sigma) = |v-v_*|^\gamma b_n(k\cdot\sigma)$, converge weakly to solutions of the Landau equation, in which the collision operator is given by

$$Q_L(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^N} a(v-v_*)[f_*\nabla f - f(\nabla f)_*]\,dv_* \right),$$

$$a(z) = \Pi(z)\Psi(|z|),$$

where $\Pi(z)$ is the projection operator defined by

$$\Pi_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2},$$

and

$$\Psi(|z|) = c_N |z|^{\gamma+2}, \tag{49}$$

with a constant c_N depending on the dimension only.

We refer to [35] (and the references therein) for a discussion of this problem in a spatially homogeneous context, including in particular the Coulomb potential in three dimensions. The much more delicate spatially inhomogeneous case is treated in [8] under very general assumptions. There again, it will be very important to gain strong compactness, and this will be a consequence of the following entropy dissipation estimate :

Proposition 4. *Assume that* (b_n) *concentrates on grazing collisions, in the sense of* (48). *Then there exists a sequence* $\alpha(n) \to 0$ *such that*

$$\begin{cases} \displaystyle\int_0^{\alpha(n)} \sin^{N-2}\theta\, b_n(\cos\theta)\,(1-\cos\theta)\,d\theta \xrightarrow[n\to\infty]{} \lambda > 0, \\ \\ \displaystyle\int_{\alpha(n)}^{\frac{\pi}{2}} \sin^{N-2}\theta\, b_n(\cos\theta)\,d\theta = \psi(n) \xrightarrow[n\to\infty]{} +\infty, \end{cases} \tag{50}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta\, b_n(\cos\theta)(\theta^2|\xi|^2 \wedge 1)\,d\theta \;\geq C\min[\psi(n), |\xi|^2]. \tag{51}$$

Proof. Define $\alpha(n)$ by the equation

$$\int_0^{\alpha(n)} b_n(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} b_n(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta.$$

Note that this quantity converges towards the positive limit $\lambda = \Lambda/2$.

We first show that $\alpha(n) \to 0$ as $n \to \infty$. Indeed, let $\alpha_0 > 0$ be an arbitrary small number, from the second equation in (48) we see that

$$\int_{\alpha_0}^{\frac{\pi}{2}} b_n(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta \xrightarrow[n\to\infty]{} 0,$$

and this entails that for n large enough, $\alpha(n) \leq \alpha_0$.

Next, we prove that $\psi(n) \to +\infty$. Let A be an arbitrary large number, and θ_0 be so small that $\Lambda/(1-\cos\theta_0) > 4A$. Since

$$\int_{\theta_0}^{\frac{\pi}{2}} b_n(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta \xrightarrow[n\to\infty]{} 0,$$

we have of course

$$\int_{\alpha(n)}^{\theta_0} b_n(\cos\theta)(1-\cos\theta)\,\sin^{N-2}\theta\,d\theta \xrightarrow[n\to\infty]{} \frac{\Lambda}{2},$$

so that for n large enough

$$\int_{\alpha(n)}^{\theta_0} b_n(\cos\theta)\,\sin^{N-2}\theta\,d\theta \geq \left(\frac{1}{1-\cos\theta_0}\right)\frac{\Lambda}{4} > A.$$

We now prove (51). For $|\xi| \geq 1$, the left-hand side in (51) is bounded below by (a multiple of)

$$|\xi|^2\int_0^{\frac{1}{|\xi|}} \sin^{N-2}\theta\, b_n(\cos\theta)\,\theta^2\,d\theta \;+ \int_{\frac{1}{|\xi|}}^{\frac{\pi}{2}} \sin^{N-2}\theta\, b_n(\cos\theta)\,d\theta\,. \tag{52}$$

We then separate two cases. If $|\xi| \leq 1/\alpha(n)$, then (for n large enough), the integral on the left of (52) is bounded below by $|\xi|^2\lambda/2$ in view of the first requirement in (50). On the other hand, if $|\xi| > 1/\alpha(n)$, then the integral on the right of (52) is bounded below by $\psi(n)$ in view of the second requirement. □

Let us give explicit examples of (50) for the two most important model cases. We use the notation

$$\zeta_n(\theta) = b_n(\cos\theta)\,\sin^{N-2}\theta.$$

- **Case 1 :**

$$\zeta_n(\theta) = n^3\zeta\,(n\,\theta)\,,$$

with ζ supported in $(0,\pi/2)$. Then for all a in the interior of the support of ζ, and $n \geq 1$,

$$\int_0^{an^{-1}} \zeta_n(\theta)\theta^2\,d\theta = \int_0^a \zeta(\theta)\,\theta^2\,d\theta \quad [\text{constant}],$$

$$\int_{an^{-1}}^{\frac{\pi}{2}} \zeta_n(\theta)\,d\theta = n^2 \int_a^{\frac{\pi}{2}} \zeta(\theta)\,d\theta.$$

- **Case 2 :**

$$\zeta_n(\theta) = \frac{1}{\log n}\zeta(\theta)1_{\theta\geq n^{-1}}, \quad \zeta(\theta) \sim \frac{1}{\theta^3} \quad \text{as } \theta \to 0.$$

Then,

$$\int_0^{n^{-1/2}} \zeta_n(\theta)\theta^2\,d\theta \sim \frac{1}{\log n}\int_{n^{-1}}^{n^{-1/2}} \frac{d\theta}{\theta} = \frac{1}{2},$$

and

$$\int_{n^{-1/2}}^{\frac{\pi}{2}} \zeta_n(\theta)\,d\theta \sim \frac{1}{\log n}\int_{n^{-1/2}}^{\frac{\pi}{2}} \frac{d\theta}{\theta^3} \sim \frac{n}{2\log n} \xrightarrow{n\to\infty} +\infty.$$

Proposition 4, together with the renormalization of cross-sections with a borderline non-integrable singularity in the relative velocity variable, like $|v-v_*|^{-N}$, will enable in [8] the first mathematical justification of the Landau approximation at small Debye cut in the spatially inhomogeneous case.

Appendix A. Fourier transform of Q^+

For the sake of completeness, we recall here the proof of Bobylev's identity (41). In fact, we shall perform here the calculation of the Fourier transform of the gain term in a general Boltzmann collision operator :

$$Q^+(g,f) = \int_{\mathbb{R}^N}\int_{S^{N-1}} B(v-v_*,\sigma)\,g'_*f'\,d\sigma\,dv_*.$$

First of all, for any test-function $\varphi(v)$, holds

$$\int_{\mathbb{R}^N} Q^+(g,f)\,\varphi(v)\,dv = \int_{\mathbb{R}^{2N}\times S^{N-1}} B(v-v_*,\sigma)g_*f\varphi'\,dv\,dv_*\,d\sigma\,.$$

Plugging $\varphi(v) = e^{-iv\cdot\xi}$ in this identity, we get

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{\mathbb{R}^{2N}\times S^{N-1}} g_*fB(v-v_*,\sigma)e^{-i\frac{v+v_*}{2}\cdot\xi}e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,dv\,dv_*\,d\sigma\,.$$

Recall that B only depends on $|v-v_*|$ and $(\frac{v-v_*}{|v-v_*|},\sigma)$. From now on, we shall display this dependence explicitly.

A key remark by Bobylev was that

$$\int_{S^{N-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) e^{-i\frac{|v-v_*|}{2}\sigma\cdot\xi}\,d\sigma$$
$$= \int_{S^{N-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-i\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,d\sigma.$$

This is a consequence of the general equality

$$\int_{S^{N-1}} F(k\cdot\sigma, \ell\cdot\sigma)\, d\sigma = \int_{S^{N-1}} F(\ell\cdot\sigma, k\cdot\sigma)\, d\sigma, \qquad |\ell| = |k| = 1$$

(due to the existence of an isometry on S^{N-1} exchanging ℓ and k).

Thus,

$$\begin{aligned}\mathcal{F}[Q^+(g,f)](\xi) &= \int_{\mathbb{R}^{2N}\times S^{N-1}} g_* f B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-i\xi\cdot\frac{v+v_*}{2}} e^{-i|\xi|\sigma\cdot\frac{v-v_*}{2}}\, dv\, dv_*\, d\sigma \\ &= \int_{\mathbb{R}^{2N}\times S^{N-1}} g_* f B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-iv\cdot\xi^+} e^{-iv_*\cdot\xi^-}\, dv\, dv_*\, d\sigma\,,\end{aligned}$$

where ξ^+ and ξ^- are defined by (39).

By the Fourier inversion formula, this is also

$$\frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}\times S^{N-1}} \Big\{ \int_{\mathbb{R}^{2N}} \hat g(\eta_*)\hat f(\eta) B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{iv_*\cdot\eta_*} e^{iv\cdot\eta} e^{-iv\cdot\xi^+} e^{-iv_*\cdot\xi^-}\, d\eta_*\, d\eta \Big\} dv\, dv_*\, d\sigma$$

$$= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}\times S^{N-1}} \hat g(\eta_*)\hat f(\eta) \left[\int_{\mathbb{R}^{2N}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{iv_*\cdot(\eta_*-\xi^-)} e^{iv\cdot(\eta-\xi^+)}\, dv\, dv_*\right] d\sigma\, d\eta\, d\eta_*\,.$$

By the change of variables $q = v - v_*$,

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{iv_*\cdot(\eta_*-\xi^-)} e^{iv\cdot(\eta-\xi^+)}\, dv\, dv_* \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} B\left(|q|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{iv\cdot(\eta_*+\eta-\xi^--\xi^+)} e^{-iq\cdot(\eta_*-\xi^-)}\, dq\, dv \\ &= (2\pi)^{N/2}\hat B\left(|\eta_*-\xi^-|, \frac{\xi}{|\xi|}\cdot\sigma\right)\delta[\eta = \xi - \eta_*],\end{aligned}$$

where δ is the Dirac measure, and $\hat B(|\xi|, \cos\theta) = \int_{\mathbb{R}^N} B(|q|,\cos\theta) e^{-iq\cdot\xi}\, dq$ denotes the Fourier transform of B in the relative velocity variable.

Thus the Fourier transform of $Q^+(g,f)$ is given by

$$\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N\times S^{N-1}} \hat g(\eta_*)\hat f(\xi-\eta_*)\hat B\left(|\eta_*-\xi^-|, \frac{\xi}{|\xi|}\cdot\sigma\right) d\eta_*\, d\sigma\,.$$

Writing $\xi_* = \eta_* - \xi^-$, we find in the end

$$\mathcal{F}[Q^+(g,f)](\xi) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N\times S^{N-1}} \hat g(\xi^-+\xi_*)\hat f(\xi^+-\xi_*)\hat B\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) d\xi_*\, d\sigma\,. \tag{53}$$

In the particular case $B(|z|,\cos\theta) = b(\cos\theta)$, we have $\hat B(|\xi_*|,\cos\theta) = (2\pi)^{N/2}\delta[\xi_* = 0] b(\cos\theta)$, and as a consequence

$$\mathcal{F}[Q^+(g,f)](\xi) = \int_{S^{N-1}} \hat g(\xi^-)\hat f(\xi^+) b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) d\sigma\,.$$

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