

PARTICLE SYSTEMS AND NONLINEAR LANDAU DAMPING

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ABSTRACT. Some works dealing with the long-time behavior of interacting particle systems are reviewed and put into perspective, with focus on the classical KAM Theory and recent results of Landau damping in the nonlinear perturbative regime, obtained in collaboration with Clément Mouhot. Analogies are discussed, as well as new qualitative insights in the theory. Finally, the connection with a more recent work on the inviscid Landau damping near the Couette shear flow, by Bedrossian and Masmoudi, is briefly discussed.

The long-time analysis of nonintegrable classical mechanical systems out of equilibrium is extremely difficult. This is a story which starts with Newton and continues to this day, still full of open problems. In this review article I shall discuss three results in this field, which will turn out to be related even though they apply to diverse physical systems. A central role will be given to Landau damping in plasma physics.

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1. SYSTEMS OF FINITELY MANY DEGREES OF FREEDOM: KAM THEORY

From the ideas of Boltzmann, Poincaré, Birkhoff and others, emerged in the first half of the 20th century the concept of **ergodicity** of a classical mechanical system: as time goes by, an ergodic system explores the phase space as much as can be done, once conservation laws and constraints are taken into account; thus it is, in a way, “as disorganized as can be”. Even though at that time none of the archetypal physically relevant mechanical systems had been shown to be ergodic, it was certainly

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a dominant view that ergodicity was a pretty general feature of mechanical systems. Let us note in passing that even for the most emblematic of those systems that are believed to be ergodic, namely the Boltzmann billiard of many identical hard spheres in a periodic box, a rigorous proof of the ergodicity property is still not available to this day (see [1] for a difficult proof of part of this famous conjecture, as well as references on the problem).

In 1954 Kolmogorov [2] shattered the hope that ergodicity would be a general feature: he showed that some systems are “organized” throughout large portions of their phase space, even though this is not imposed by conservation laws. This was the start of the so-called Kolmogorov–Arnold–Moser (KAM) Theory, which received crucial inputs from Arnold, Moser, Herman and others.

In their study of Hamiltonian dynamics, Kolmogorov and his followers focused on perturbations of integrable systems, for at least two reasons:

- a perturbative approach allows to exploit some of the structure of integrable systems;
- the most celebrated long-time stability problem in mathematical physics, namely the stability of the Solar system, has traditionally been considered as a perturbative problem, because of the smallness of the ratio of the mass of planets to the mass of the Sun.

Indeed, the Hamiltonian of the Solar system (idealized, fully Newtonian, with point planets, and no satellites, neither external stars or other celestial bodies) can be written, up to a multiplicative factor ε , as

$$H(X, Y) = \sum_{1 \leq i \leq N} \left(\frac{|Y_i|^2}{2\mu_i} - \frac{\mathcal{G} \mu_i m_0^i}{|X_i|} \right) + \varepsilon \sum_{1 \leq i < j \leq n} \left(\frac{Y_i \cdot Y_j}{m_0} - \frac{\mathcal{G} M_i M_j}{|X_i - X_j|} \right). \quad (1)$$

Here m_0 , x_0 and p_0 stand for the mass, position and momentum of the Sun, and m_i , x_i , p_i for the mass, position and momentum of the planet i ; all the planets are numbered from 1 to N ; $\varepsilon = (\sum_{i \neq 0} m_i)/m_0$, $M_i = m_i/\varepsilon$, $m_0^i = m_0 + m_i$, $\mu_i = m_0 m_i / [\varepsilon(m_0 + m_i)]$, and the variables are $X_i = x_i - x_0$ and $Y_i = p_i/\varepsilon$; moreover it is assumed that $\sum p_i = 0$. See for instance [3, Section 6] for a discussion of this system, which was studied intensively from Laplace to Poincaré.

If one sets $\varepsilon = 0$ in (1) then the motions of the planets decouple into *independent* motions, orbiting around the Sun; then for $\varepsilon > 0$ small enough (in the “real” Solar system, ε is of the order of 10^{-3}), the problem arises whether small perturbations may build up constructively, leading to a major turnover in the behavior of the system. Lagrange, Laplace and Gauss had shown that in the absence of resonances

between trajectories, the Solar system remains stable over very large times $O(1/\varepsilon^2)$, but left open the problem of stability over even longer times.

Thus Kolmogorov studied general Hamiltonian systems of the form $H_\varepsilon(x, p) = H_0(p) + \varepsilon H_1(x, p)$. Considering the frequency vector $\omega = (\omega_1, \dots, \omega_N) = \nabla H_0(p)$, Kolmogorov reinforced the nonresonance condition

$$\forall k = (k_1, \dots, k_N) \in \mathbb{Z}^N \setminus \{0\}, \quad k \cdot \omega \neq 0$$

into the Diophantine condition

$$\forall k \in \mathbb{Z}^N \setminus \{0\}, \quad |k \cdot \omega| \geq \frac{\kappa}{|k|^\gamma}, \quad (2)$$

where $\kappa, \gamma > 0$; and he showed that under certain simple assumptions on the Hamiltonian, for ε small enough, trajectories corresponding to the frequency vector ω are just distorted versions of corresponding trajectories for the Hamiltonian H_0 . Since (2) is satisfied by almost all frequency vectors ω for $\gamma > N - 1$, as long as κ can be reduced, Kolmogorov deduced that the major part of the phase space is filled up with invariant low-dimensional submanifolds (which have the topology of a torus).

Even though Kolmogorov assumed a nondegeneracy condition which excluded the case of the Solar system (the latter was treated only in 2004 by Féjóz [3], building up on ideas by Arnold and Herman), his theoretical results had a huge impact. Here is one of the many versions of Kolmogorov's theorem:

Theorem 1. *(adapted from Chierchia [4]) For $(x, p) \in \mathbb{T}^N \times \mathbb{R}^N$ let $H_0 = H_0(p)$ and $H_1 = H_1(x, p)$ be analytic functions, and $H_\varepsilon = H_0 + \varepsilon H_1$. Let \mathcal{V} be a bounded open set in \mathbb{R}^N ; assume that the Hessian $\nabla^2 H_0(p)$ is nondegenerate for $p \in \mathcal{V}$. Then for $\varepsilon > 0$ small enough there are a measurable set $\mathcal{V}_\varepsilon \subset \mathcal{V}$, and a change of variables $(x, p) \rightarrow (x', p')$, defined on $\mathbb{T}^N \times \mathcal{V}_\varepsilon$, which smoothly maps trajectories of the Hamiltonian system defined by H_0 into trajectories of the integrable Hamiltonian system defined by H_ε . Moreover, $|\mathcal{V} \setminus \mathcal{V}_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0$.*

In other words, locally, most of the trajectories of H_ε are very regular, since they are deformations of well-chosen trajectories for H_0 . (Note: $(x, p) \rightarrow (x', p')$ is not a canonical change of variables, but rather, for a given p_0 , $x \rightarrow (x'(x, p_0), p'(x, p_0))$ is the trace, on the subspace $(p = p_0)$, of a canonical change of variables.) While Kolmogorov's conclusion negated the ergodicity, it only applied to a very perturbative regime; in practice the bounds required by KAM theory are rarely met in physically realistic systems, and certainly not in our Solar system. However, looking back at the field now, one can say that both ergodic theory and KAM theory have had a lasting impact on our understanding of classical mechanical systems, even though

only a tiny number of physically realistic systems have been shown to belong to either one or the other theory.

Two interesting features of KAM theory are worth pointing at for the forthcoming discussion. First, even though it deals with dynamical systems, the core of the theory is the construction of a smooth solution to a PDE problem of Hamilton–Jacobi type, a typical instance of which would be

$$H_\varepsilon(x, \nabla\Phi + c) = \bar{h},$$

where $c \in \mathbb{R}^N$ and $\bar{h} \in \mathbb{R}$ are well-chosen parameters, and Φ is the unknown real-valued function. The second remark is that KAM theory encodes the quantitative non-resonance bound (2) into a property of regularity loss for the solution of the linearized equation

$$\omega \cdot \nabla\varphi = f, \tag{3}$$

where φ is the unknown and f is a source term: indeed, from (2) and the Fourier transform of (3), we see that γ counts the number of derivatives which are lost in the process of reconstructing φ from f . (See the Appendix for reminders about regularity and regularity loss.) This makes it possible to apply the methods developed by Kolmogorov, Nash, Arnold, Moser and Nirenberg to handle partial differential problems with finite loss of derivatives in a perturbative setting: solve the problem by a Newton approximation scheme or a variant thereof, and

- either work à la Kolmogorov, with analytic norms defined with a convergence radius (or a related parameter) which shrinks slightly at each step of the scheme;
- or work à la Nash–Moser, with high enough C^r regularity, using regularization operators to re-create the regularity which is lost at each step of the Newton scheme.

In both cases, the very fast convergence of the Newton scheme is most useful to absorb the very large constants created by the loss of regularity in each iteration step.

2. STATISTICAL LIMITS

By considering the initial configuration of the system in a probabilistic sense, KAM Theory introduced statistics in an otherwise purely deterministic problem, giving up the idea to make definite predictions. This is consistent with chaos theory and the phenomenon of sensitivity to initial conditions, discovered by Maxwell, Poincaré and Hadamard, which often prevents any effective prediction of the final fate of a physical system.

Statistics can also be introduced with a different purpose: as a way to describe the state of the system, when it is made of so many constituents (stars in a galaxy,

electrons in a plasma, etc.) that an individual tracking of these objects is impossible. Then the unknown in the model is the statistical distribution $f(t, x, v)$ of particles in the phase space of positions x and velocities v at time t . (The reference measure in phase space is the Liouville measure $dx dv$.)

This statistical approach has been extremely successful for practical applications and theoretical studies, with two archetypal continuous models: the Boltzmann equation, when interactions occur on a scale that is negligible compared to the spatial scale of reference; and the Vlasov equation, when both scales are comparable.

In its simplest form, the Boltzmann equation is obtained from Newtonian dynamics in the Boltzmann–Grad limit, that is, when the number of particles becomes extremely large while the total cross-section remains of order unity. It reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f),$$

where Q is the Boltzmann collision operator, which is localized in space-time, taking into account the effect of collisions of particles; these collisions are classically assumed to be elastic, that is, energy-preserving. This model has been successfully used for gases, especially in a dilute regime, and more recently adapted to quantum and granular (inelastic) interactions; see [5, 6, 7, 8] for an overview of the whole theory.

In contrast, the Vlasov equation is obtained through the mean-field limit, that is, when each particle feels the combined effect of all other particles; it reads

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \\ F[f] = -\nabla W *_x \int f dv, \end{cases} \quad (4)$$

where $F[f](t, x)$ is the force created at time t and position x by the interaction potential W and the mass distribution f . This model is a cornerstone of plasma physics [9], and also galactic dynamics [10], even though the statistical approximation is somewhat more debatable in the latter context.

Before going on, let us note that neither the Boltzmann equation nor the Vlasov equation have been rigorously derived in the large; in particular, Lanford’s celebrated validity theorem for the Boltzmann equation was only established in short time, or for a rare cloud dispersing in the whole space [7, 11, 12]; and all validity theorems for the Vlasov equation assume that the interaction at small scales is either smooth [13, 14, 15] or not too singular [16]. Even though the physical relevance of these “validity theorems” may be debated (for instance, neglecting quantum effects in the derivation of the classical Boltzmann equation is probably unrealistic), they should

be considered as some of the most fundamental theoretical problems in statistical mechanics.

3. BOLTZMANN'S H THEOREM

Discovered around 1872 by Boltzmann, the H Theorem is a milestone in nonequilibrium statistical mechanics. A genuinely nonlinear result, it shows that for the classical (elastic!) Boltzmann equation, under the action of uncorrelated collisions, the Boltzmann entropy

$$S = - \iint f \log f \, dv \, dx$$

cannot decrease in time. Not only does this provide a guideline for the long-time behavior of the Boltzmann equation, but it also identifies a fundamental driving force in the statistical world: **entropy increase**. In particular, for classical particles, it predicts the homogeneous Gaussian asymptotic behavior of the probability distribution function. It also provides a theoretical argument in support of equilibrium statistical mechanics, when the processes at work are microscopically reversible and macroscopically irreversible.

While the entropy increase is easy to prove formally, its control depends on a subtle interplay between information theoretical considerations and fluid mechanics. Much progress has been done over the past twenty years in the mathematical understanding of the H Theorem in the Boltzmann equation, so that it seems fair to say that the only essential missing piece in the puzzle is the understanding of the regularity of solutions to the Boltzmann equation; see [17, 18] for surveys and explanations.

4. LANDAU'S DAMPING

After Boltzmann's H Theorem, Landau's damping is arguably the second main insight which we have into the large-time behavior of classical statistical clouds of particles. In contrast with the H Theorem, it is principally a linear result. To state it, let us work for instance in the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, and linearize the Vlasov equation near a spatially homogeneous equilibrium $f^0(v)$: if h stands for the perturbation, then the linearized equation is

$$\partial_t h + v \cdot \nabla_x h + F[h] \cdot \nabla_v f^0(v) = 0. \quad (5)$$

Let $\rho^1(t, x) = \int h(t, x, v) \, dv$ be the linearized spatial density perturbation. For any spatial frequency vector $k \in \mathbb{Z}^d$, the mode of order k , $\hat{\rho}^1(t, k) = \int \rho^1(t, x) e^{-2i\pi k \cdot x} \, dx$,

satisfies a closed equation:

$$\widehat{\rho}^1(t, k) = \widetilde{h}_i(k, kt) + \int_0^t K_k^0(t - \tau) \widehat{\rho}^1(\tau, k) d\tau, \quad (6)$$

where $h_i(x, v) = h(0, x, v)$ is the initial value of the perturbation, and $\widetilde{h}_i(k, \eta) = \iint h_i(x, v) e^{-2i\pi k \cdot x} e^{-2i\pi \eta \cdot v} dx dv$ is its position-velocity Fourier transform. Moreover, the kernel $K_k^0(t)$ is defined by the formula

$$K_k^0(t) = -4\pi^2 \widehat{W}(k) \widetilde{f}^0(kt) |k|^2 t,$$

where \widetilde{f}^0 is the kinetic Fourier transform of the equilibrium f^0 and \widehat{W} is the spatial Fourier transform of the interaction potential W .

The remarkable feature of (6) is that all spatial modes of ρ^1 *decouple*: in this sense the linearized Vlasov equation is a completely integrable system. Then the stability of that equation can be investigated by classical recipes for the Volterra equation, based on the analysis of the Laplace transform

$$(K_k^0)^L(\lambda) = \int_0^\infty K_k^0(t) e^{2\pi \lambda^* t} dt, \quad \lambda \in \mathbb{C}.$$

If the initial disturbance h_i is analytic, then the source term $\widetilde{h}_i(k, kt)$ in (6) decays like $O(e^{-\Lambda|k|t})$ for some $\Lambda > 0$; for this decay to be preserved by (6), one needs to impose that $(K_k^0)^L \neq 1$ for all λ with $\Re \lambda \leq \Lambda|k|$. Assuming that f^0 is analytic, this condition is satisfied (for some $\Lambda > 0$) if and only if the **Penrose stability criterion** holds true. Here is the most general version of that stability condition: for any $k \neq 0$, let f_k^0 be the one-dimensional marginal of f^0 on k (obtained by integrating out along directions orthogonal to k : $\forall w \in \mathbb{R}, f_k^0(w) = \int_{k^\perp} f^0(wk + z) dz$); then it is required that for any $\omega \in \mathbb{R}$

$$(f_k^0)'(\omega) = 0 \implies \widehat{W}(k) \int_{\mathbb{R}} \frac{(f_k^0)'(v)}{v - \omega} dv < 1. \quad (7)$$

Most of the time, one does not use the full version of this criterion, but one of several simplified sufficient conditions: in plasma physics ($\widehat{W}(k) = \text{cst.}/|k|^2$), one typically assumes that f^0 is radially symmetric, or depends only on one variable and has only one critical point (necessarily a maximum) with respect to that variable; in galactic dynamics ($\widehat{W}(k) = -\text{cst.}/|k|^2$), one usually assumes that the distribution is a Gaussian whose ratio T/ρ (=temperature/density) is high enough to meet the classical Jeans stability criterion [10]. But any of these popular criteria is a particular case of the stability criterion (7), which was devised by Penrose through a neat complex variable argument.

Summarizing the results of this analytic computation: If the data are analytic, and the Penrose stability condition holds, then for the linearized Vlasov equation, (i) all nonzero spatial modes of ρ^1 converge to 0; (ii) the force converges to 0 also. The convergence is like $O(e^{-2\pi\Lambda|k|t})$ at frequency k , for some $\Lambda > 0$; that is, higher frequencies decay faster. This is the Landau damping. One can also check that in this linearized approximation, the kinetic density perturbation h has a definite limit:

$$h(t, x, v) \xrightarrow[t \rightarrow \pm\infty]{} \langle h_i(\cdot, v) \rangle_x \equiv \int h_i(x, v) dx \quad \text{weakly.}$$

Here “weak convergence”, sometimes also called “convergence in the mean”, is defined as the convergence of Fourier modes, or equivalently of smooth observables; that is, $\int h(t, x, v)\varphi(x, v) dx dv$ converges to $\int h_i(y, v)\varphi(x, v) dx dy dv$ for any given smooth test function φ ; but the distribution function h itself oscillates wildly in large time and does not converge in the usual sense.

Finally, the analysis carries through for data which are regular but not analytic: for instance, it is sufficient to impose that $\nabla^3 f^0$ and ∇h_i are integrable; then all the previous conclusions will remain true, except that the time decay will only be like an inverse power. See for instance [19, Section 3] for precise statements.

Even though Landau’s analysis was only linear, its influence was considerable: it suggested that a homogenization mechanism is operating in a mean-field system, even when the entropy remains constant. What is happening in this “entropy-preserving relaxation mechanism” is that mixing trajectories generate fast kinetic oscillations which globally compensate each other in the velocity-averaging procedure leading from the distribution function to the force field. One can show in particular that $h(t, x, v)$ displays kinetic oscillations whose frequency is of order $O(1/t)$ as $t \rightarrow \infty$; more precisely, of order $O(1/|k|t)$ at spatial frequency k .

It has been widely speculated after Landau’s contribution that entropy-preserving relaxation mechanisms occur in a more general setting, possibly far from equilibrium. Another important consequence is the possibility for a mixing, slowly diffusing system to relax much faster than could be guessed from the study of entropy production, or more generally the rate of diffusive processes. This is the paradigm of **violent relaxation**.

5. NONLINEAR LANDAU DAMPING

The linearized Vlasov equation (5) was obtained as an approximation of the nonlinear Vlasov equation (4), but this does not guarantee that the asymptotic behavior of the linear Vlasov equation is an approximation of the asymptotic behavior of the

nonlinear Vlasov equation. (Think of $u(t) = 1 + \varepsilon t$: although the equation reduces to $u(t) = 1$ in the limit $\varepsilon \rightarrow 0$, the large-time limit of the equation is $+\infty$ for any given $\varepsilon > 0$.) In fact there are several a priori reasons to doubt that the study of the linearized equation gives any hint on the long-time behavior of the nonlinear equation, which can be rewritten

$$\partial_t h + v \cdot \nabla_x h + F[h] \cdot \nabla_v f^0 = -F[h] \cdot \nabla_v h. \quad (8)$$

For a start, due to fast oscillations of the distribution function, $\|\nabla_v h\|$ diverges linearly in time for any “reasonable” choice of norm; so the quadratic term $F[h] \cdot \nabla_v h$ (which is neglected in the linearization) is expected to be much larger, after some time, than the term $F[h] \cdot \nabla_v f^0$ (which is retained in the linearization); this basic objection was raised by Backus [20] in 1960. A second objection to linearization is the fact that conservation laws are quite different for both equations: in the linearized setting, all position averages $\int h(x, v) dx$, are preserved, while in the nonlinear setting, all nonlinear integrals $\iint C(f(x, v)) dx dv$ are constant. A third objection is that linearization has killed the only term involving a velocity-derivative $\nabla_v h$, and as a general rule dominant order terms often have an important influence on the qualitative behavior of a partial differential equation.

In spite of all these reservations, Mouhot and I proved [19] that the nonlinear Vlasov equation does enjoy the same long-time mixing properties as the linearized equation, when the initial datum is a perturbation of a stable equilibrium. This theorem depends on two regularity assumptions which are not essential in the linearized analysis, but do play a crucial role in our nonlinear analysis. The first one is that the interaction potential be no more singular than Poisson: $\widehat{W}(k) = O(1/|k|^2)$ for $k \in \mathbb{Z}^d$. The second one is that the data should be analytic, or at least of Gevrey regularity \mathcal{G}^ν , for ν small enough. We recall that Gevrey regularity corresponds to a *fractional exponential decay* in Fourier space; reminders about this notion are in the Appendix. All this is summarized in the next result.

Theorem 2. (from Mouhot–Villani [19]) *Let $d \geq 1$ and let $W : \mathbb{T}^d \rightarrow \mathbb{R}$ be a periodic interaction potential such that $\widehat{W}(k) = O(1/|k|^{1+\gamma})$, $\gamma \geq 1$. Let $f^0 = f^0(v)$ be an analytic function of $v \in \mathbb{R}^d$, such that $(f^0)^{(n)}(v) = O(C^n n!)$ and $\widetilde{f}^0(\eta) = O(e^{-2\pi\lambda_0|\eta|})$ for some constants $C > 0$, λ_0 and for all n and η . Assume that f^0 satisfies the generalized Penrose stability condition. Let $h_i(x, v)$ be a small initial perturbation which is well-localized in frequency space and in velocity space:*

$$|\widetilde{h}_i(k, \eta)| \leq \varepsilon e^{-c(|k|+|\eta|)}; \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} |h_i(x, v)| e^{\beta|v|} dv dx \leq \varepsilon, \quad (9)$$

for some constants $c, \beta > 0$. Then for ε small enough, the solution of the nonlinear Vlasov equation with potential W and initial datum $f^0 + h_i$ satisfies

$$f(t, x, v) \xrightarrow[t \rightarrow \pm\infty]{} f_{\pm\infty}(v) \text{ weakly}; \quad F(t, x) \xrightarrow[t \rightarrow \pm\infty]{} 0 \text{ strongly},$$

where the convergence is exponentially fast. Moreover, the same conclusions remain true if the first condition in (9) is replaced by $|\tilde{h}_i(k, \eta)| \leq \varepsilon e^{-c(|k|+|\eta|)^\alpha}$ with $\alpha > 1/(\gamma + 2)$, except that the rate of convergence is now $O(e^{-K|t|^\alpha})$.

Here $f_{+\infty}$ and $f_{-\infty}$ are smooth limit homogeneous profiles which in general are distinct. We note that in our work [19] this result is proven only for α close to 1; but the more general statement can be deduced from the methods there and an adaptation of the computations.

Some of the main arguments in the intricate proof of Theorem 2 involve

1) *a reinterpretation of the damping property into a regularity property*: Instead of focusing on the mysterious property of decay of the force, we prove that $f(t, x, v)$ oscillates, in the long run, just as wildly as the solution of free transport $\partial_t f + v \cdot \nabla_x f = 0$. These wild oscillations should happen in a very “ordered” way: composing back to free transport should lead to something very smooth. In other words, $f(t, x + tv, v)$ should be uniformly smooth as $t \rightarrow \infty$, or equivalently the bulk of the Fourier transform at spatial mode k should be located around the time-dependent kinetic mode $-kt$; a property which we call “gliding regularity”. Then the gliding regularity implies decay by mixing, and the time-decay of the force field can be read in the decay of the solution in Fourier space.

Moreover, the uniform smoothness of $f(t, x + tv, v)$ implies not only the decay of the force, but also the *weak convergence* of $f(t, x, v)$ to a homogeneous stationary state as $t \rightarrow \infty$.

2) *a careful study of the self-induced echoes*: Recall the famous Malmberg–Wharton **echo experiment** from the sixties: a plasma which is initially at rest (homogeneous, stable) will react to a sequence of two electric pulses, one at time t_0 with spatial frequency ℓ , and one at time $t_0 + \tau$ with spatial frequency $k - \ell$, by a spontaneous “echo” occurring at time

$$t_e = t_0 + \left(\frac{k - \ell}{k} \right) \tau,$$

assuming that k and ℓ are colinear and pointing into opposite directions. The analysis of that effect is usually performed in a linear approximation [21, Section 7.3], but of course it should also be taken into account in the nonlinear analysis: inside the plasma the various modes all interact with each other, so that many

echoes are to be expected at all times, and the problem is to show that these echoes do not accumulate constructively up to a massive response. Note that the strength of coupling between modes k and ℓ is proportional to $\widehat{W}(k-\ell)$: so the decay assumption on \widehat{W} is important to control the interaction of far away modes.

Following the regularity interpretation, a way to evaluate the strength and evolution of these echoes is to study the linear equation

$$\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v \bar{f}(t, x, v) = 0, \quad (10)$$

in which $\bar{f}(t, x, v) = f^0(v) + \bar{h}(t, x, v)$ is not stationary and only assumed to satisfy a uniform gliding regularity bound in large time. We call this the response analysis: thinking of $\bar{f}(t, x, v)$ as a background distribution on which the distribution function f acts, how much is f affected by the additional response $-F[f] \cdot \nabla_v \bar{h}$? This influence affects the evolution equation for $\rho = \int h dv$, and the Fourier modes of ρ are not decoupled any longer.

When we were considering only the linearized approximation (5), then we had an integral equation (6), which can be written compactly into the form $\rho - K^0 * \rho = S(k, kt)$; but in (10) the time-varying part of \bar{f} will lead to additional terms. Taking these into account, one can establish a new integral inequality:

$$\|\rho - K^0 * \rho\| \lesssim S(k, kt) + \int_0^t K(t, \tau) \|\rho(\tau)\| d\tau,$$

where the time-kernel $K(t, \tau)$ takes into account possible echoes, and the influence at time t of impulsions from past times τ . It can be shown that $K(t, \tau)$ is all the larger as W is rougher: this reflects the fact that the nonlinear sensitivity of the problem depends on the *singularity of the interaction W at small scales*. As already noticed, this is in contrast with the linear analysis, in which the singularity at small scales plays basically no role.

A complicated, rather precise estimate can be given for the kernel $K(t, \tau)$ [19, Section 7], but the important fact is that it tends to concentrate at times τ which are in rational proportion to t . If the interaction potential is not too singular, the concentration is not too bad for $\tau \simeq t$, so that something remains of the *delay* which is associated with the echo phenomenon; and this is good news, because delay in a time evolution equation implies an additional stability. To appreciate the effect of time delay on simplified model problems, compare the behavior of a function u which satisfies the simple equation $u(t) = A + \int_0^t K(t, \tau) u(\tau) d\tau$, in the typical cases $K(t, \tau) = 1$ (exponential growth), $K(t, \tau) = t \delta(\tau - t)$ (arbitrarily wild growth) and $K(t, \tau) = t \delta(\tau - t/2)$ (slow superpolynomial growth, like $t^{c \log t}$).

3) *a positive loop between the trajectorial analysis and the force decay*: Fast damping implies that particle trajectories are asymptotic to those of free transport; nearly free transport implies good mixing properties; and good mixing implies fast damping. These features reinforce each other, as can be made explicit by a technical iteration scheme.

6. NOVEL MATHEMATICAL ANALYSIS OF THE DAMPING

Let us briefly describe three mathematical ingredients which allow to exploit the ideas of the previous section:

- We introduce a family of specific analytic norms, which is a mixture of two well-known algebra norms for analytic functions: denoting by \mathbb{N}_0^d the set of (n_1, \dots, n_d) with each n_i a nonnegative integer, we set

$$\|f\| = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} e^{2\pi\mu|k|} (1 + |k|)^s \frac{n!}{\lambda^n} \|(\nabla_v + 2i\pi\tau k)^n \widehat{f}(k, v)\|_{L^p(dv)}.$$

Here $\lambda > 0$ quantifies the analytic regularity in the v variable; $\mu > 0$ quantifies the analytic regularity in the x variable; $s \geq 0$ is a finite-regularity correction in the position variable; p is a Lebesgue integrability exponent; and $1/\tau$ is a frequency, which can be thought of as the expected frequency of kinetic oscillations. These norms enjoy interesting properties, and are well adapted to the dynamics of kinetic equations. Despite their complexity, they can be compared with very simple norms by means of certain functional inequalities; so they do not appear in the theorem itself, only in its proof.

- The delay which survives from the echo analysis can be quantified by means of a **regularity loss**, which is best appreciated by doing estimates on the amplitude of the spatial Fourier modes. While the actual equations for this are a bit complicated, we can write up a toy model which captures the essence of these estimates: if we introduce $\rho = \int f dv$, $\varphi_k(t) = |\widehat{\rho}(t, k)|$, keep only the strongest interactions between neighboring frequencies, localize the echoes in time to the extreme by pretending they are instantaneous, and replace particle trajectories by free transport, we end up with an infinite system of estimates of the form

$$\varphi_k(t) \leq S(kt) + \frac{ct}{(k+1)^{\gamma+1}} \varphi_{k+1} \left(\frac{kt}{k+1} \right), \quad (11)$$

in which γ is the exponent of decay of the force $-\nabla W$ in Fourier space (so that $|\widehat{W}(k)| = O(1/|k|^{1+\gamma})$). Think of (11) as a linear estimate in which $S(kt)$ is a source and (φ_k) is the unknown: then one can obtain an estimate on (φ_k) in which the

regularity loss manifests itself through multiplication of the bound on S by a factor $\exp((a|k|t)^{\frac{1}{\gamma+2}})$, $a > 0$. This *fractional exponential* regularity loss mathematically quantifies how bad the accumulation of echoes can be, and makes the link with the singularity of the interaction potential. It also expresses the fact that (11), and probably (10) as well, loses an infinite number of derivatives.

- In this perturbative context, the loss of regularity in the resolvent equation can be caught up by a **Newton scheme**, as Kolmogorov initially did for KAM Theory. Then the loss of regularity determines the critical regularity in which this scheme of proof will work: that is, Gevrey- $(\gamma + 2)$. For the physically relevant case of a Newton or Coulomb interaction ($\gamma = 1$), this is Gevrey-3 smoothness. This is in contrast with KAM theory, in which C^r smoothness can be handled thanks to the finite loss of regularity.

More details can be found in my lecture notes on Landau damping [21].

7. BRIDGING UP KAM AND LANDAU DAMPING

It is well-known that KAM Theory goes away in infinite dimension, in the sense that all estimates vanish rapidly when the dimension increases to infinity. Still, the discussion in the previous sections provides a kind of dictionary, with similarities and differences, between KAM Theory and perturbative nonlinear Landau damping. This is summarized in Table 1.

8. INCOMPRESSIBLE SHEAR FLOW

One of the oldest mathematical topics in perturbative hydrodynamics is the stability of two-dimensional shear flows. As an archetype, consider the Couette (linear) shear flow, in which the velocity is horizontal and in linear relation with the depth, say $U_0(x, y) = (y, 0)$; then perturb this into $(y + u(x, y), v(x, y))$, where u and v are very small initially. Will the resulting flow remain close to the Couette flow for all times? Will it relax in large time to a shear flow?

Such issues were studied as early as the 1880's by Kelvin, Rayleigh, Reynolds and Sommerfeld. While the Couette flow is linearly stable (in the sense of the evolution of Fourier modes), it is usually found to be experimentally unstable, a phenomenon sometimes called the Sommerfeld paradox. Note that the linearized Euler equation, written in terms of the perturbed vorticity $\omega = \partial_x v - \partial_y u$, takes the form $\partial_t \omega + y \partial_x \omega = 0$, so that it is formally similar to a kinetic equation with no interaction potential.

TABLE 1. Similarities and differences between KAM Theory and non-linear Landau damping

<u>KAM theory</u>	<u>nonlinear Landau damping</u>
classical mechanics, N bodies	classical mechanics, $N = \infty$
$\varepsilon > 0$ measures strength of coupling	$\varepsilon > 0$ measures strength of nonlinearity
$\varepsilon = 0$: completely integrable system	$\varepsilon = 0$: “completely integrable” linearized Vlasov
statistical choice of initial conditions	statistical description of particles
Diophantine condition implies stability	Smoothness implies stability
resonances between trajectories due to particle interactions	time resonances (echoes) due to wave interactions
finite regularity loss	infinite regularity loss
C^k critical regularity	Gevrey critical regularity
can be handled by Newton scheme	can be handled by Newton scheme
by Kolmogorov or Nash–Moser methods	by Kolmogorov’s method only

In 1907 Orr [22] gave the first thorough analysis of this linear stability, pointing out that it is subtle, since (i) the vorticity becomes very much filamented as time becomes large; (ii) the fast oscillations of the vorticity come together with the convergence of the velocity field; (iii) this convergence cannot be faster than $O(1/t, 1/t^2)$ (the vertical component relaxes faster than the horizontal one); (iv) there is no conservation of the kinetic energy in the limit, so that the problem is linearly unstable in the kinetic energy norm; (v) the stability is fragile, because high-frequency modes become enhanced one after another in large time. Indeed, the (x, y) -Fourier transform of the velocity is proportional to $\tilde{\omega}_i(k, \eta - kt)$, which may be much larger than $\tilde{\omega}_i(k, \eta)$, especially when $t \simeq \eta/k$.

Thus the picture described by Orr for the evolution of $\omega(x, y)$ is rather close to the one discovered forty years later by Landau for the evolution of a kinetic perturbation $h(x, v)$ in a plasma. One often speaks of “inviscid damping” for the mixing of the vorticity, and this has been studied for more general flows [23]. Orr wondered whether, in spite of the slow convergence, the Couette flow would be nonlinearly stable for very small perturbations.

In the linear approximation, it is simpler to study the mixing, damping and stability for the Couette flow than for the Vlasov equation; but in the nonlinear setting, on the contrary, the Couette flow situation turns out to be the more tricky of the

two problems. First, because of the slow convergence, trajectorial properties of the flow in large time are much worse (especially because $1/t$ is not time-integrable). Secondly, any shear flow $(u(y), 0)$ will generate a specific dynamics, so the asymptotic behavior of particle trajectories cannot be universal, and will depend on the initial velocity profile.

In 2013, Bedrossian and Masmoudi [24] adapted the ideas from our study of the nonlinear Landau damping [19] and introduced new ingredients to prove the following result, confirming for the first time the analysis of Orr:

Theorem 3. (from Bedrossian–Masmoudi [24]) *Consider the horizontally periodic 2-dimensional nonlinear Euler equation ($x \in \mathbb{T}, y \in \mathbb{R}$) in the form*

$$\partial_t \omega + y \partial_x \omega + \xi \cdot \nabla \omega = 0, \quad \xi = \nabla^\perp \Delta^{-1} \omega,$$

where $\Omega = -1 + \omega$ is the total vorticity, and $U(t, x, y) = (y, 0) + \xi(t, x, y)$ is the velocity field. If the initial perturbation $\xi_i(x, y) = \xi(0, x, y)$ decays fast enough as $|y| \rightarrow \infty$ and is small enough in the Gevrey space \mathcal{G}^ν , for some $\nu < 2$, then the flow U converges to a shear flow in large time:

$$U(t, x, y) \xrightarrow[t \rightarrow +\infty]{} U_\infty(x, y) = (y + u_\infty(y), 0).$$

This Theorem, like Theorem 2, applies in an analytic setting, but eventually relies on a Gevrey regularity; note that the amount of regularity which is required is even higher, since now the critical Gevrey exponent in the proof is 2, instead of 3 for the Vlasov–Poisson equation in Theorem 2. The exponent 2 can also be determined from the (infinite) regularity loss in an auxiliary problem. Moreover, the convergence of the vorticity can only be obtained in a set of moving coordinates which depend on the solution itself:

$$\left\| \omega \left(t, x + t(y + u_\infty(y)) + \Psi(t, y), y \right) - \omega_\infty(x, y) \right\| = O \left(\frac{1}{t} \right) \quad \text{as } t \rightarrow +\infty,$$

where $\Psi(t, y) = O(\|\xi_i\|^2 (\log t)^2)$ is a higher order perturbation of controlled growth. The norms used in the proof are even more complex than those in our work [19], since their mere definition depends on the solution. These norms are constructed by solving an auxiliary toy model where only the worst case resonances are taken into account. As a final remark, in [24] the recourse to a Newton scheme is circumvented by the use of a carefully defined time-dependent norm, plus technical paraproduct estimates to control the regularity of bilinear terms.

9. CONCLUSIONS AND SPECULATIONS

In this review paper I have discussed how three classical stability problems set in very different situations are actually related in a strong way. Regularity is at the same time a unifying element in these problems, and a way to uncover some of their differences.

The main results from Mouhot–Villani [19] and Bedrossian–Masmoudi [24] establish nonlinear Landau damping for Vlasov and 2-dimensional Euler equation, respectively, near appropriate stable equilibria, for all times, under periodic boundary conditions and in analytic or high enough Gevrey regularity. Our results [19] also solve a problem which, although simpler looking, was open for long, namely the nonlinear stability of linearly stable (not necessarily monotonic in $|v|$) homogeneous equilibria of the Vlasov equation. Furthermore, comparison of these two works shows that Landau damping in Vlasov–Poisson and in 2-dimensional Euler equation, even though they rely on similar properties (mixing, control of echoes, gliding regularity), also exhibit distinct features and seem to be associated with distinct critical regularities.

Still, this analysis opens up even more questions. One of them is whether the Gevrey regularity is indeed critical for nonlinear Landau damping to hold, or whether this is just a feature of the proofs. Another one is the status of regularity: Landau damping seems indeed very much dependent on regularity, and it was actually shown by Lin & Zeng [25] that there is no hope for damping in the Vlasov–Poisson model when the initial datum is allowed to explore a neighborhood of the equilibrium in a topology involving less than $3/2$ derivatives; very similar results have also been obtained for the 2-dimensional Euler equation [26]. Of course “most” distributions should definitely be nonsmooth, so this is probably not yet the end of the story concerning the relation between regularity and stability.

Another issue is whether the parallel between KAM theory and nonlinear Landau damping can be deepened. Observe already a major difference between both settings: while in KAM Theory there always remains an exceptional portion of phase space (corresponding to $\mathcal{V} \setminus \mathcal{V}_\varepsilon$ in the notation of Theorem 1) which is not fibered by stable trajectories, in the context of Landau damping we can prove the stability in a whole neighborhood of the equilibrium; this can be attributed to the fact that the core of the KAM problem is the solution of a stationary PDE, while the Landau damping problem rests on an evolution equation, in which resonances do occur but are time-averaged. It can also be related to the fact that Landau damping is an infinite-dimensional problem, so there is no obstruction preventing a whole neighborhood

of the equilibrium to be filled with homoclinic or heteroclinic trajectories, provided this neighborhood is defined in a strong enough topology.

In the context of KAM theory, it is known that very interesting phenomena, like Arnold diffusion, can occur in the region of phase space which lies outside the invariant tori: is there anything like this in Landau damping theory, and does it have anything to do with the regularity?

Finally, of course, our proof of nonlinear Landau damping [19] is strongly related to the homogeneity assumption and the perturbative nature of the problem. This leaves completely open the generalization to inhomogeneous equilibria (such as BGK equilibria), and, even more difficult, to wilder perturbations (violent relaxation analysis). While Landau damping has been used to motivate the concept of violent relaxation (since both rely on mixing), it is also conceivable that Landau damping applies only to a very regular, KAM-like part of data, while violent relaxation would correspond to a completely different regime, conjecturally ergodic in some sense.

APPENDIX: REGULARITY

The regularity of a real-valued function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is an expression of how smooth its variations are. The most common ways of measuring the regularity of a function are by looking at bounds on its successive derivatives $f^{(n)}$, where n is a multi-index; or by looking at how fast its Fourier transform \widehat{f} decays at high frequencies. Let us recall some basic facts about this, considering only periodic functions for simplicity (so that the frequency space will be equal to \mathbb{Z}^d , and Fourier transform will be just Fourier series).

The most popular notions of regularity are:

- C^r regularity: f is of class C^r if all its derivatives up to order r are continuous;
- H^s (Sobolev) regularity: f is of class H^s if $\sum_k |\widehat{f}(k)|^2 (1 + |k|^2)^s < +\infty$;
- C^∞ regularity: f is of class C^∞ if it is of class C^r for all $r > 0$, or equivalently of class H^s for all $s > 0$;
- C^ω (analytic) regularity: f is of class C^ω if its derivatives satisfy $f^{(n)} = O(n!/r^n)$ for some constant $r > 0$ independent of the multi-index n ; or equivalently, if $\widehat{f}(k) = O(e^{-2\pi r|k|})$ for some constant $c > 0$ independent of the frequency k . (Then the radius of convergence of the Taylor series of f is at least r .)

In the case of analytic regularity, many norms can be used to quantify the regularity; two interesting choices are $\|f\|_\lambda := \sum_n (\lambda^n/n!) \sup |f^{(n)}|$ and $\|f\|_\lambda := \sum_k e^{2\pi\lambda|k|} |\widehat{f}(k)|$, which depend on a parameter $\lambda > 0$ playing the role of a radius of convergence. Both are algebra norms, which means that $\|fg\|_\lambda \leq \|f\|_\lambda \|g\|_\lambda$.

Suppose that one is given a linear equation like $Lf = g$: knowing the regularity of g , what can be said of the regularity of f ? Of course the answer depends of the type of the equation: if it is elliptic, then typically second derivatives of f have the same regularity as g , etc. In the case then f is a priori γ degrees less smooth than g , one says that the equation *loses* γ derivatives. Since differentiation corresponds to multiplication by $2i\pi k$ in Fourier, the loss of γ derivatives can be seen in Fourier space: it means that $(|\widehat{f}(k)|)_{k \in \mathbb{Z}^d}$ is controlled in a suitable sense by $(|k|^\gamma |\widehat{g}(k)|)_{k \in \mathbb{Z}^d}$.

Between the C^∞ and C^ω regularity lies the Gevrey regularity: pick up an index $\nu \geq 1$, then f is \mathcal{G}^ν if it satisfies $f^{(n)} = O(C^n n!^\nu)$ for some $C > 0$ (independently of n), or equivalently if $\widehat{f}(k) = O(e^{-c|k|^{1/\nu}})$ for some $c > 0$ (independently of k). Gevrey functions are extremely smooth, yet for $\nu > 1$ they are not restricted by all the rigidity attached to analyticity: for instance, a Gevrey function may very well vanish in an open set and not be identically zero.

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