MATHEMATICS OF GRANULAR MATERIALS

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ABSTRACT. This is a short and somewhat informal review on the most mathematical parts of the kinetic theory of granular media, intended for physicists and for mathematicians outside the field.

CONTENTS

INTRODUCTION

Granular materials are a very trendy subject nowadays, and the number of publications devoted to it has grown tremendously since the beginning of the nineties. These contributions deal with experiments, modelling, numerical simulations, industrial design as well as theoretical work. Some of the most spectacular effects appearing in the dynamics of granular gases are reviewed in a short and pedagogic survey by Barrat, Trizac and Ernst [3]; they include clustering, spontaneous loss of homogeneity, inverse Maxwell Demons, modification of Fourier's law, violation of equipartition of energy, and non-Gaussian equilibrium kinetic distributions. There is also a recent textbook on the subject by Brilliantov and Pöschel [18].

This field constitutes a potential whole new area of applications opening up for mathematicians; yet the relevant mathematical literature is still restricted, due to the extreme theoretical complexity of the subject. The present survey deals with one of the (relatively) most advanced parts of the theory, in which kinetic models are used for granular gases, and interactions are described by inelastic collisions. On this subject, two short reviews by mathematicians are already available in the published literature: the first one is a concise and very clear introduction by Cercignani [25]; the other one was written by the author a few years ago [44, Chapter 5, Section 2]. While these references might still be acceptable from the point of view of modelling or the presentation, they are by now obsolete as far as the results are concerned; this is not surprising since the subject is still very young (the first truly mathematical paper about granular collisions is arguably the work by Benedetto, Caglioti and

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Pulvirenti [7], as late as 1997). Here I shall endeavor to fill this gap by presenting a tentative up-to-date review of rigorous results in inelastic collisional kinetic theory. The style will be somewhat informal to ensure that the text can be read by a wide audience; more precise results and statements can be found in the quoted research papers, including the two papers by Mischler, Mouhot and Rodriguez Ricard [39, 40] in the present volume.

Although the body of available physics literature is enormous, I decided to keep the bibliography to the minimum, quoting almost only mathematically oriented papers, most of them in direct relation to the subject, with the main exception of a few review papers like [3]; the interested reader will have no trouble finding physical documentation by starting with the references there. For the classical kinetic theory, the reader will find almost everything that he or she needs in the above-mentioned review [44]. Also I did not address hydrodynamic limits (see e.g. [34, 43, 10, 18]) which are still poorly understood from the mathematical point of view, and somewhat controversial from the physical point of view.

Acknowledgements: This set of notes is an expanded version of a course which I gave in February 2005 in Institut Henri Poincaré, on the invitation of Alain Barrat, in a thematic semester about granular material. Many thanks are due to Alain and the other participants for their invitation and their active participation, which contributed in the presentation of these notes. Additional thanks are due to Clément Mouhot for helpful discussions during the preparation of the courses, and to Sasha Bobylev, José Antonio Carrillo, Irene Gamba and Giuseppe Toscani for their helpful comments on a preliminary version of this text. These thanks extend to the anonymous referees for their careful reading and comments.

Dedication: This review paper is dedicated to the memory of Frédéric Poupaud, one of the most inventive specialists of kinetic theory in recent years, equally at ease in theory and modelling. Frédéric explored many areas of physics, from quite pure to quite messy, with the eyes of kinetic theory. His untimely death is a heavy loss for our community and for science in general.

1. Modelling

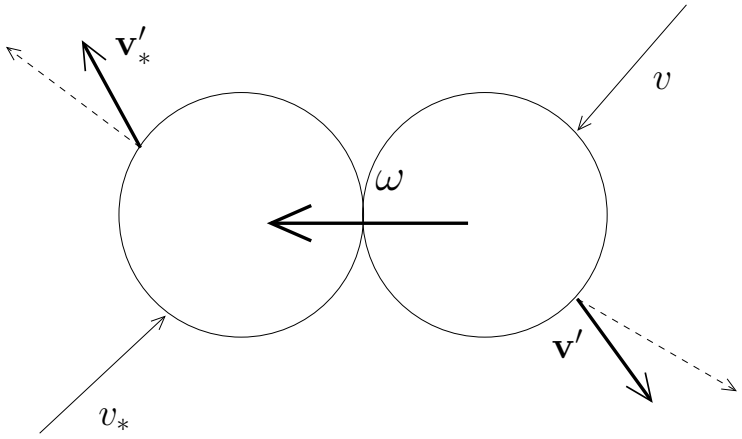
A typical kinetic model for granular material takes the following form: the unknown is a time-dependent distribution function in phase space $f(t, x, v)$ (t is time, x is position and v is velocity) satisfying an equation like

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_v \cdot (F f) = \mathcal{C}(f) \quad + \text{ diffusion and/or friction terms.} \tag{1}$$

Here $v \cdot \nabla_x$ is the usual transport operator, F is a force, that may depend on t and x, or even on v, and $\mathcal{C}$ is an **inelastic collision operator** describing the effect of collisions with energy dissipation built in (this energy dissipation might be due to the roughness of the surface or just to a non-perfect restitution, and does not affect the conservation of momentum). It is natural to assume that we are working in a 3-dimensional space. I shall not discuss boundary conditions (which are very tricky), but this issue seems to be quite important in the field, since in experiments granular materials are rarely left alone, but usually forced in one or another way (shaking, etc.)

The particles themselves are considered as small balls, just as in the popular model of hard spheres. The usual rules of kinetic description apply: for instance, one might define a temperature in terms of the variance of the velocity distribution.

1.1. **Collisions.** As said above, collisions are supposed to incorporate inelasticity. I shall only consider inelasticity due to an imperfect restitution of energy, and neglect rotational degrees of freedom, although they might be quite important [34, 25]. The illustration below provides a schematic picture of what goes on. The incoming velocities are v and v_*; ω is the impact direction. Would the collision be elastic, the outgoing velocities would be given by the dashed arrows below; but because of inelasticity effect, there is some loss of momentum in the impact direction, resulting in the boldface arrows indicating the outgoing velocities v' and v'_*.



Let e stand for the **restitution coefficient** or (in)elasticity parameter:

$$\langle v' - v'_*, \omega \rangle = -e \, \langle v - v_*, \omega \rangle, \qquad 0 \leq e \leq 1.$$

Then the collision equations can be solved into

$$\text{(2)} \qquad \begin{cases} v' = v - \dfrac{1+e}{2} \, \langle v - v_*, \omega \rangle \, \omega, \\ \\ v'_* = v_* + \dfrac{1+e}{2} \, \langle v - v_*, \omega \rangle \, \omega. \end{cases}$$

In particular the variation of kinetic energy is

$$\frac{|v'|^2}{2} + \frac{|v'_*|^2}{2} - \frac{|v|^2}{2} - \frac{|v_*|^2}{2} = - \left(\frac{1-e^2}{4} \right) \langle v - v_*, \omega \rangle^2 \leq 0.$$

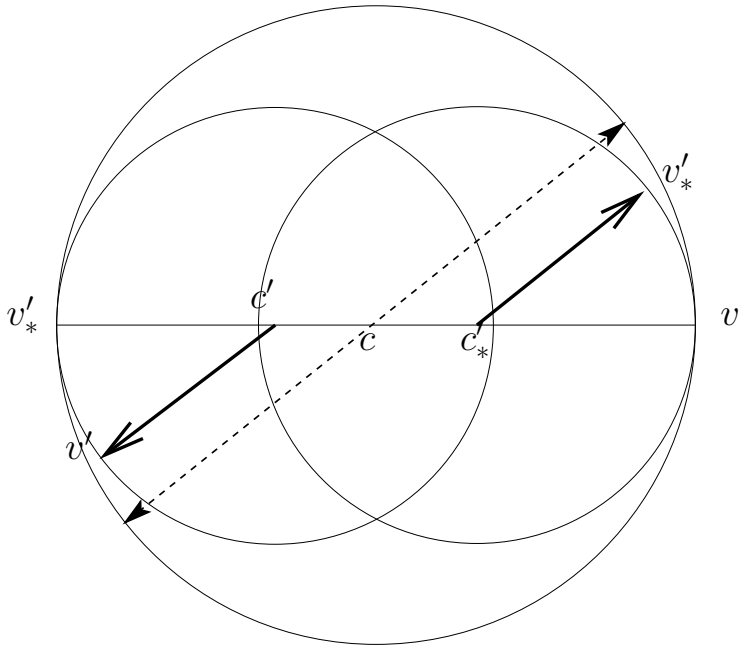
In general the coefficient e might depend on the norm of the relative velocity, that is $|v - v_*|$, on the deviation angle θ; for phenomenological reasons a constitutive dependence on the temperature T of the gas is also often assumed. For some authors, this dependence has important consequences (see e.g. [11]), for others it does not matter so much; many researchers work with a constant restitution coefficient. It is good to keep in mind that the two limit regimes $e = 1$ and $e = 0$ respectively correspond to **elastic** (no loss of energy) and **sticky** collisions (after collision, particles travel together).

The velocities v' have to lie on a certain sphere S with center c' and radius $r \leq |v - v_*|/2$. It is often convenient to parameterize collisions by the direction σ of

the vector $v' - c'$; in the terminology of [44], this is the σ-representation. Here are the corresponding formulas:

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \left(\dfrac{1-e}{4}\right)(v - v_*) + \left(\dfrac{1+e}{4}\right)|v - v_*|\,\sigma \\ \\ v'_* = \dfrac{v + v_*}{2} - \left(\dfrac{1-e}{4}\right)(v - v_*) - \left(\dfrac{1+e}{4}\right)|v - v_*|\,\sigma. \end{cases} \tag{3}$$

The parameter σ varies in the unit sphere S^2. The reader can meditate on the figure below, which is drawn in velocity space (in dashed lines are the elastic rules of collision).



To write the Boltzmann equation, one needs to compute the **pre-collisional** velocities: Given two velocities v and v_* after collision, find the velocities ${}'v$ and ${}'v_*$ before collision. Here are the formulas:

$$\begin{cases} {}'v = \dfrac{v + v_*}{2} - \left(\dfrac{1-e}{4e}\right)(v - v_*) + \left(\dfrac{1+e}{4e}\right)|v - v_*|\sigma \\ \\ {}'v_* = \dfrac{v + v_*}{2} + \left(\dfrac{1-e}{4e}\right)(v - v_*) - \left(\dfrac{1+e}{4e}\right)|v - v_*|\sigma. \end{cases} \tag{4}$$

Note carefully that ${}'v$ and ${}'v_*$ do **not** coincide with v' and v'_*; in other words, the collisions are not reversible. This is in agreement with the fact that we are observing a dissipative process. Accordingly, the Jacobian of the transformation $(v, v_*) \to ({}'v, {}'v_*)$ is not 1, but

$$J = \frac{|v - v_*|}{e^2\,|{}'v - {}'v_*|}. \tag{5}$$

Remark. In dimension 1, elastic collisions are "stupid": since $\{v', v'_*\} = \{v, v_*\}$ (velocities are either preserved or swapped), these collisions do not change the velocity distribution. It is not so for inelastic collisions:

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*) \right\},$$

depending on the value of $\sigma \in \{-1, +1\}$. Similarly,

$$\{'v, 'v_*\} = \{v, v_*\} \quad \text{or} \quad \left\{ \frac{v + v_*}{2} \pm \frac{1}{2e}(v - v_*) \right\}.$$

1.2. **Collision operator.** Two collision operators are classically used for describing the effect of collisions on the distribution function. The first one is the **Boltzmann collision operator**:

$$Q_B(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, |v - v_*| \Big(J \frac{|'v - 'v_*|}{|v - v_*|}\, f('v)\, f('v_*) - f(v)\, f(v_*) \Big). \tag{6}$$

Here the factor $|v - v_*|$ is characteristic of the "hard sphere model": the mean number of collisions between particles of given velocities is proportional to the difference of velocities (this is intuitively obvious); the factor J is given by (5). The parameters t and x do not appear explicitly in (6), because this operator only acts on the velocity dependence: in other words, collisions are supposed to be **localized** in space and time. The fact that the Boltzmann equation is written in terms of the "local tensor product" $f(x,\cdot) \otimes f(x,\cdot)$ reflects the fundamental statistical assumption of *molecular chaos* (no correlation between pre-collisional velocities).

Notation. It is customary to write f' as a shorthand for $f(v')$ or $f(x, v')$; similar rules apply to f, f_*, f'_*, $'f$, $'f_*$. Also the Boltzmann operator naturally splits into two parts: $Q(f,f) = Q^+(f,f) - Q^-(f,f)$, which are traditionally called the **gain term** and the **loss term**. The notation which I use is by no means universal: all kinds of symbols are used to denote pre- or post-collisional velocities, e.g. v^*, v^{**}, etc. I recommend the convention with v' and $'v$ (due to V. Panferov).

The second popular model is the **Enskog collision operator**:

$$\begin{aligned} Q_E(f,f) = r^2 \int_{\mathbb{R}^3} dv_* \int_{S^2} d\sigma\, |v - v_*| \Big(J \frac{|'v - 'v_*|}{|v - v_*|}\, G(x, x - r\omega)\, f(x, 'v)\, f(x - r\omega, 'v_*) \\ - G(x, x + r\omega)\, f(x,v)\, f(x + r\omega, v_*) \Big). \end{aligned} \tag{7}$$

Here $r > 0$ is the radius of the particles, ω is the impact direction, and $G(x,y)$ is the *correlation function* between points x and y. There are obvious similarities and obvious differences between the Enskog and the Boltzmann collision operator: in particular, the space variable enters explicitly in equation (7), which means that collisions are **delocalized** in space. Moreover, the assumption of molecular chaos has been dropped, and now the function G takes care of correlations. Roughly speaking, G is defined by the formula

$$f^{(2)}(x, v; y, w) = G(x,y) f(x,v) f(y,w),$$

where $f^{(2)}$ is the two-particle distribution density. The need for such a correlation function is more or less obvious if one realizes that the assumption of "thick" spheres

implies correlations between various positions: as a trivial example, if a particle is present at position x, then no particle can be present at y if $|x-y| < r$. In practise, G might be given by an explicit expression of x and y, or an expression of x, y and $\rho(x)$, $\rho(y)$, where $\rho(x)$ stands for the density of the gas at x; it might even depend on the whole density profile. For more information see the short but intricate enough discussion in Cercignani [25].

The mathematical status of the Boltzmann and the Enskog collision operators are not comparable. The localization in the Boltzmann equation leads to formal simplicity and analytical difficulty; it is also at the source of a rich mathematical structure, which has contributed to establish the Boltzmann equation as one of the most renowned and challenging partial differential equations studied by mathematicians. The Enskog equation on the other hand leads to a very messy theory and horrendous calculations; it is often considered with a bit of awe by physicists and mathematicians. It is important to note that, while the "validity" of the Boltzmann equation for hard spheres has been established (even though only under certain restrictions, see [44, Chapter 1, Section 2.1]) by Lanford, no such result exists for the Enskog equation. Thus the latter model might be considered as a heuristic or phenomenological model.

In the context of granular material, however, the Enskog equation is supposed to be much better adapted, since particles are almost "macroscopic" in size. Whether this will be sufficient to re-boost the theory of the Enskog equation is still unclear. For the moment, no serious mathematical work on the inelastic Enskog equation has been performed, so all the rest of these notes will deal with the Boltzmann equation. Note that the distinction between both models is irrelevant when the space dependence is taken away.

As a final general comment, it is very striking to see the devastating effects of the inelasticity assumption on the classical theory of the Boltzmann equation.

1.3. **When to use pre- and post-collisional velocities.** The pre-collisional velocities $'v$ and $'v_*$ appear in the collision operator and are therefore useful whenever one wants to study properties of that operator; in particular regularity issues, that might be used in a regularity study for the solution of the equation.

However, most of the time, in practical applications, one studies the density distribution f via observables, that are integrals of f against some test function which depends only on v, or on x and v. In that case the correct equation is the *weak* version of the Boltzmann collision operator:

$$\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)\,dv = \int |v-v_*| f(v) f(v_*) \left(\varphi(v') - \varphi(v)\right) dv\, dv_*\, d\sigma. \tag{8}$$

In that sense the formulas for post-collisional velocities are used much more often, in practise, than the formulas for pre-collisional velocities.

1.4. **Variants and simplifications.** The chief simplification is **spatial homogeneity**, which amounts to look for solutions of the form $f(t,v)$. While this is a huge simplification, it often allows a fine description of purely kinetic effects in complicated subjects.

Another way to simplify the Boltzmann equation consists in a **reduction of the dimension**, either by symmetry or phenomenological arguments. Thus one may obtain two-dimensional or even one-dimensional models. As I mentioned above, a

one-dimensional elastic Boltzmann equation is meaningless, but a one-dimensional inelastic one makes sense.

It is possible to replace the collision kernel $|v-v_*|$ by $|v-v_*|^\gamma$, for some exponent γ which varies between, say, -1 and 1. In classical (elastic) kinetic theory, γ can range from -3 (Coulomb interaction) to 1 (hard spheres). There is no real microscopic basis for such a generalization in the context of granular material, but there might be phenomenological reasons. A particularly interesting case for analytical resolution is $\gamma = 0$ (Maxwellian collisions).

Finally, one may take into account only *quasi-frontal* collisions ($\theta \simeq \pi$, or, what amounts to the same by symmetry, $\theta \simeq 0$, that is grazing collisions) or *quasi-elastic* collisions ($e \simeq 1$). In one dimension both asymptotic regimes are about the same. It is sometimes found in physics literature that velocities in a granular gas have a tendency to align on each other, which might be a justification for the use of models where grazing collisions play an important role. Some models that can be used in this respect look like

$$Q_L + \nabla_v \cdot \Big(f\, \nabla_v (f * |v|^{\gamma+2} v) \Big),$$

where Q_L is a Landau-Fokker-Planck (elastic) collision operator. Here $*$ stands for convolution with respect to the velocity; since $*$ commutes with ∇_v, the operator on the right-hand side really is proportional to

$$\nabla_v \cdot \left(\int_{\mathbb{R}^3} f f_* |v - v_*|^\gamma (v - v_*)\, dv_* \right).$$

There is nothing mysterious in the exponent $\gamma + 2$: Taking into account the two derivatives in v, it corresponds to a homogeneity like $|v|^\gamma$ ($\gamma = 1$ for hard spheres). See Li and Toscani [37] for a study of how certain qualitative properties of this equation vary with γ.

Here is a "historically" important example of oversimplified model: at the beginning of the nineties, McNamara and Young suggested the following collision-type operator in dimension 1:

$$Q_{\mathrm{MNY}}(f,f) = \frac{\partial}{\partial v} \Big(f (f * |v| v) \Big),$$

which remained popular for some time. Most of the first works by mathematicians in granular media, in particular those by Pulvirenti and co-authors between 1997 and 2000, dealt with the McNamara-Young model.

1.5. **Other operators.** Apart from collisions, one may add to the kinetic model various terms which either model external physical forces, or arise from particular situations. Here are four such possibilities.

(i) **Heat bath**: Many experiments about granular material include shaking, as a way to input energy into the system, counterbalancing the freezing due to energy loss. A rather trivial but seemingly not so absurd model consists in a heat bath, or white noise forcing: this amounts to adding to the right-hand side of the kinetic equation a term like

$$T_e\, \Delta_v f,$$

where T_e is an "external temperature", normalized to 1 in the sequel. Such models were introduced by mathematicians and physicists in various contexts [46, 6, 11].

(ii) **Friction**: Particles may experience extra friction forces if they are going through a viscous fluid or something to that effect. Here again, there is a trivial model consisting in adding a drift term to the equation; the most simple case being that of a linear drift,

$$\alpha \nabla_v \cdot (fv) \qquad (\alpha > 0).$$

(iii) **Shear flow**: The behavior of grains in a flowing fluid is of interest. Cercignani [26] proposed the following simplified model. Look for solutions of the free Boltzmann equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_B(f, f)$$

between two surfaces $x_2 = 0$ and $x_2 = L$, moving at respective speed 0 and V, with bounce-back boundary conditions. Make the ansatz $f(t, x, v) = f(t, c)$, where $c = v - u(t, x)$ is the deviation to the local mean velocity; and assume some self-similarity in the solution, in particular $u(t, x) = K(t)x + v_0(t)$. Then it is found that a sufficient condition for $f(t, c)$ to solve the problem is

$$\frac{\partial f}{\partial t} - \nabla_c f \cdot (Kc) = Q(f, f), \tag{9}$$

where K is a matrix that can be written, in a well-chosen coordinate system, with only one nonzero entry, outside the diagonal. In practise, this means that a simple solution to the shear flow problem can be constructed by solving the spatially homogeneous Boltzmann equation with an additional term like, say,

$$v_1 \frac{\partial f}{\partial v_2}$$

on the right-hand side.

(iv) **Homogeneous Cooling State**: Solutions of the "free" inelastic Boltzmann equation will typically lose energy until all the particles travel at the same speed (at each point x). For simplicity, let us consider a spatially homogeneous situation and assume that the mean velocity is 0; then the asymptotic state is δ_0 (all particles at rest). To study more precisely the asymptotic behavior, it is natural to "zoom" on the velocity distribution close to 0, that is to rescale the distribution function. The Boltzmann collision operator does satisfy some scaling properties: if the collision kernel is proportional to $|v - v_*|$ in dimension d, then

$$Q\big(f(\lambda \cdot), f(\lambda \cdot)\big) = \lambda^{-(d+1)} Q(f, f)(\lambda \cdot).$$

It is not difficult to find that the new unknown $\tilde{f}(\tilde{t}, \tilde{v})$ defined by

$$f(t, v) = (1 + t)^d \, \tilde{f}\Big(\log(1 + t), (1 + t)v\Big)$$

satisfies a nice equation:

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} = Q(\tilde{f}, \tilde{f}) - \nabla_{\tilde{v}} \cdot (\tilde{v} \tilde{f}).$$

That equation now may admit a nontrivial steady state, because the energy-dissipative effects of the Boltzmann collision operator are balanced by the energy input coming from the term in $-\nabla_v \cdot (v \cdot f)$. Any result about the convergence to such a steady state leads to some qualitative information about the way the original equation converges

to δ_0. Of course, to a steady state in the new variables $\tilde{t} = \log(1+t)$, $\tilde{v} = (1+t)v$ is associated a *self-similar solution* in the original variables t, v, traditionally called "homogeneous cooling state". The moral here is that, to study the relaxation to homogeneous cooling states, one is led to consider the Boltzmann equation with an additional **anti-drift** term

$$-\nabla_v \cdot (fv).$$

Note that the change of variable from time t to time $\tilde{t}$ is *logarithmic*, so one has to be careful in interpreting these results (a fast decay in the $\tilde{t}$ scale does not mean so fast a decay in the original scale).

By the way, similar "logarithmic" rescalings of variables are classical in the literature about the large-time behavior of diffusion equations, and also lead to added drift terms. For instance, this rescaling transforms the heat equation $\partial_t f = \Delta_v f$ into the linear Fokker-Planck equation $\partial_t f = \Delta_v f + \nabla_v \cdot (fv)$. (Note that here we gain a drift, not an anti-drift!) The literature related to such changes of variables is so vast that I prefer not to provide any reference.

1.6. **"Simple" problems.** Here is a non-exhaustive list of problems that at least can be formulated in a relatively precise mathematical way.

- What is the behavior of temperature in a granular gas? In a spatially homogeneous gas, a simple dimensional analysis leads to the intuitive formula

$$\frac{dT}{dt} \simeq -\text{const.}\, T^{3/2}$$

(think that $|v - v_*|$ has the same physical dimension as $\sqrt{T}$). This suggests that the temperature decays like $O(t^{-2})$ with time; this is classically called **Haff's law**, since it was proposed by Haff [33] in the beginning eighties. Among other things, Haff's law implies that the solution at time t lies at distance $O(t^{-1})$ to the Dirac mass δ_u, where u is the mean velocity. In a spatially inhomogeneous context, things might become more intricate and the temperature may undergo important spatial variations.

- What is the large-time behavior? In the case of elastic collisions, a lot of work has been devoted to the approach of Maxwellian equilibrium, in relation with the maximum entropy principle and Boltzmann's H Theorem. For inelastic collisions there is no H Theorem (only an approximate version of it when the restitution parameter e is close to 1) and no intuitive way to attack the problem.

- What about the large velocity behavior? Is it still Gaussian, as in the elastic case, or not? We shall see that the answer is negative.

- Inhomogeneities and clustering: In the classical theory, spatial homogeneity is a very strong assumption, but at least it is *stable*: under appropriate boundary and size conditions, a weakly inhomogeneous gas remains weakly inhomogeneous for all times. In the case of inelastic collisions, there is extensive numerical evidence to support the possibility of inelastic collapse (temperature falls down to 0 at certain places, meaning that the variance of the distribution vanishes), clustering (aggregates of particles with zero temperature form) and ensuing pattern formation. From a theoretical perspective, it would be very exciting if such behavior could be demonstrated; for that, new tools probably have to be developed.

To summarize the state of the art about these questions, I could say that several (very) simplified models are quite well understood; and that as a general rule the spatially homogeneous theory is starting to be in a quite decent situation, although still incomplete. On the other hand there is essentially nothing relevant known about inhomogeneities.

1.7. **Heuristics about the tail behavior.** One of the most famous predictions of inelastic kinetic theory is the possibility of *overpopulated distribution tails*, meaning that the typical large-velocity behavior of the velocity distribution is not Gaussian, but displays much thicker tails (so the inelasticity results in the presence of many very fast particles, which is not so intuitive). In addition to the fact that it contradicts the universality of the familiar Gaussian distribution, one of the reasons why this prediction is famous is certainly the fact that it can be obtained by quite simple heuristics.

Here is one way to argue. For the free Boltzmann equation, the asymptotic state is the Dirac mass, say at zero velocity. Thus the gain term (which increases the energy) is dominated in some sense by the loss term (which decreases the energy). Hence, when studying the large-velocity behavior, it is natural to neglect the former in front of the latter. Then the loss term is asymptotically proportional to something like $K|v|^\gamma f$, where K is some constant, if the collision kernel is proportional to $|v-v_*|^\gamma$ ($\gamma = 1$ for hard spheres). Let us look for a radially symmetric distribution function $f(v) = f(r)$, $r = |v|$, and make the ansatz $f(r) \simeq e^{-ar^\beta}$. Then $Q^-(f,f) \simeq Kr^\gamma e^{-ar^\beta}$; $\Delta_v f = r^{-2}\partial_r(r^2\partial_r f) \simeq Cr^{2\beta-2}e^{-ar^\beta}$; $\nabla_v \cdot (fv) \simeq r^\beta e^{-ar^\beta}$. By balancing dominant terms in the equation for the steady state one easily obtains

$$\begin{cases} \text{heat bath:} & \beta = 1 + \frac{\gamma}{2}; \\ \text{heat bath + friction:} & \beta = 2; \\ \text{anti-drift:} & \beta = \gamma. \end{cases}$$

This argument predicts for the hard spheres model tails like $e^{-a|v|^{3/2}}$ for the heat bath, like $e^{-a|v|^2}$ for heat bath and friction, and like $e^{-a|v|}$ for homogeneous cooling states. Such behaviors were predicted, on the basis of slightly more precise arguments, by Ernst and Brito (see the review papers [29, 28, 27]; e.g. [28, Section 4.4]). There is all reason to believe that this is the correct theoretical answer, let apart the special case $\beta = \gamma = 0$ which is more subtle (as discussed in [28]).

The shear flow case is more intricate since there is no reason for solutions to be radially symmetric.

1.8. **What is the trouble with non-Gaussianity?** The reader might legitimately ask: Elastic collisions lead to Gaussian distributions, but inelastic collisions do not, so what is the big deal? It is no wonder that a change in the assumptions leads to a change in the conclusion...

Yet there is some reason to be struck by the non-Gaussian behavior. In fact, in the classical kinetic theory there are three different, yet related, ways to arrive at the Gaussian, or rather the Maxwellian distribution (i.e. a Gaussian with scalar covariance matrix). The first and probably most well-known is the one by Boltzmann, which can be stated informally as follows:

Let f be a probability distribution such that

$$f(v)f(v_*) = F\big(v + v_*, |v|^2 + |v_*|^2\big).$$

Then f is Maxwellian.

This argument obviously deals with *pairs* of particles; it is related with the chaos assumption (particles are uncorrelated before collision) and the conservation laws (elasticity means conservation of $v + v_*$ and $|v|^2 + |v_*|^2$ through a collision).

Now comes another argument by Maxwell, which considers the system of all particles as a whole:

Consider N particles with velocities $v_1, \dots, v_N$. Assume that these velocities are randomly uniformly distributed, given the only constraints that $(v_1 + \dots + v_N)/N = u$ and $(|v_1 - u|^2 + \dots + |v_N - u|^2)/N = T$ are determined. Then, the law of one these particles, say v_1, is very close (if N is large) to a Maxwellian distribution with mean u and temperature T.

The requirement that the velocities should be uniformly distributed apart from the constraints coming from the conservation laws can be seen as a most elementary example of *microcanonical ensemble* in statistical mechanics. Actually Maxwell's theorem is probably the most basic example of equivalence of ensembles.

The last statement, also due to Maxwell (before Boltzmann's argument) is the one which is of most direct interest here. It goes as follows:

Let v be a random velocity in $\mathbb{R}^3$ such that (i) the distribution of v is radially symmetric, i.e. it only depends on $|v|$, (ii) the coordinates of v are independent random variables; then the distribution of v is Maxwellian.

In other words, if we have probability distributions f_i, $i = 1, 2, 3$, such that $f_1(v_1)f_2(v_2)f_3(v_3) = g(v_1^2 + v_2^2 + v_3^2)$, then each f_i has to be Gaussian. (The dimension 3 has nothing special; any dimension $d \geq 2$ would work just the same.)

The converse of the last statement may be more striking. If the velocity distribution is not Gaussian, then there are correlations between the components of the velocity distribution. Roughly speaking, by measuring v_1 we can obtain some information about v_2. This looks quite strange if we are used to classical gases, but from the point of view of mathematics there is no contradiction with the isotropicity assumption: take for instance an extreme case where the probability distribution is supported in $\{|v| \leq R\}$. If I measure v_1 to be greater than $\sqrt{R^2 - 1}$, then I know for sure that $v_2^2 + v_3^2$ is less than 1; so the various components are indeed correlated.

2. Maxwellian toolbox

2.1. **The model.** Maxwell collisions are defined by the assumption that the collision kernel is a function of just the deviation angle θ: so the weak form of the collision kernel is

$$\int_{\mathbb{R}^3} Q_B(f, f)\,\varphi = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} b(\cos\theta) f f_*(\varphi' - \varphi)\, dv\, dv_*\, d\sigma,$$

and similarly the restitution coefficient e only depends on θ. Actually it is common practise to choose b and e to be just constants.

This modification makes the model analytically much simpler. To understand why, think that, by integrating out successively σ, then v_* in the loss term one now obtains $Q^-(f, f) = K\rho f$, where K is a constant and ρ is the macroscopic density

(which can be taken to be just 1 in the spatially homogeneous case). This should be compared with the loss term for hard spheres, that is $Kf(f * |v|)$.

In classical kinetic theory, Maxwell collisions may be interpreted as describing interactions between particles that repel each other with a force proportional to the inverse of the fifth power of the distance. For the description of granular material, such an interpretation is certainly irrelevant, so one should rather consider the model as just an analytical simplification. An alternative point of view consists in forgetting about the "true" collisional mechanism and replace it by *random dynamics*: Whenever two particles collide, the parameter σ is chosen randomly on S^2 (uniformly if b is a constant). This interpretation is due to Kac (see [44, Chapter 1, Section 2.1] for the elastic case).

With the added structure in the Maxwell model come new specific tools and features which may be useful. The most important are:

- closed moment equations;
- Fourier transform;
- contracting distances;
- information-theoretical tools.

All this is described and explained, in the elastic case, in [44, Chapter 4]. In fact there is a good analogy between the Boltzmann collision operator with Maxwellian kernel on one hand, and the usual convolution operators on the other hand. I should insist that so far, all this extra structure has been exploited only in the spatially homogeneous case, which is the only one considered in this section.

2.2. **Temperature dependence.** Replacing the collision kernel $|v - v_*|$ by a function of just θ leads to irrelevant temperature behavior. To keep a correct temperature law it is better to replace $|v - v_*|$ by something which has the same physical dimension: an *averaged* value of the relative velocity. The most common choice is something like the *square root of the temperature* T: indeed, if v and v_* are independent random variables distributed according to the probability density $f(v)$, then the mean value of $|v - v_*|^2$ is

$$\int f f_* |v - v_*|^2 \, dv \, dv_* = 2 \left(\int f(v) |v|^2 \, dv - \left| \int f(v) v \, dv \right|^2 \right) = 2T.$$

So it would be natural to replace $|v - v_*|$ by just $\sqrt{2T}$. There is a dimensional constant in front of the collision operator, so the $\sqrt{2}$ does not really matter, but the fact that the collision rate is proportional to the square root of the temperature is important.

With this modification, from the weak formulation of the Boltzmann equation it is easy to check that (i) the mean momentum $\int f \, v \, dv$ is preserved, (ii) the temperature $T(t)$ at time t satisfies the differential equation

$$\frac{dT}{dt} = -KT^{3/2},$$

which integrates exactly to

$$T(t) = \frac{T(0)}{(1 + at)^2} \qquad (a > 0),$$

where a is proportional to $e(1-e)$ for a constant coefficient e. Hence Haff's law is correct for this model (this is of course not surprising, we kind of enforced it). Thus, in the spatially homogeneous case, the effect of the temperature dependence comes only through an explicit function of time in front of the collision kernel, and the analytical simplicity of the model is preserved.

2.3. **References.** The three-dimensional inelastic Maxwellian model was first introduced and studied by Bobylev, Carrillo and Gamba [12]; at almost the same time, simple one-dimensional models were studied independently by Baldassari, Marconi and Puglisi [2]; see also Ben Naïm and Krapivski [4, 35]. Since then these equations have been studied thoroughly by several authors. Good recent review papers have been written by Ben Naïm and Krapivsky [5], Ernst and Brito [28, 27], who survey the existing physical literature and physical results. In [28] the emphasis is put on the Maxwell model as a model arising from stochastic dynamics.

Apart from the above-mentioned works, another important source on the Maxwell model is the series of papers by Bobylev and Cercignani, sometimes with other coauthors such as Carrillo, Gamba or Toscani, in various issues of the *Journal of Statistical Physics* [13, 14, 15, 17].

2.4. **Moments equations.** Let $P(v)$ be a polynomial in v, that is, in the variables v_1, v_2, v_3. To compute the time-evolution of the observable associated with P, one should evaluate

$$\int_{\mathbb{R}^3} Q^+(f,f)\,P\,dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} f f_* \left(\int_{S^2} P(v')\,d\sigma \right) dv\,dv_*.$$

The expression inside inner parentheses depends only on the polynomial, and has to satisfy certain homogeneity conditions. Actually, one can show that it is a polynomial in v and v_*; in particular it can be *split* into a sum of products of polynomials:

$$\int_{S^2} P(v')\,d\sigma = \sum_i P_i(v) P_{*,i}(v_*).$$

Then one can integrate against $f f_*\,dv\,dv_*$ and separate variables, obtaining products like $(\int f P_i)(\int f P_{*,i})$. Of course these polynomials P_i, $P_{*,i}$ are of *lower* degree than P itself. The conclusion is that the equation for the evolution of a polynomial moment of a certain order can be expressed in terms of polynomial moments of lower order. Since the first moments can be evaluated explicitly, we understand that *in principle, the time-evolution of all polynomial moments can be exactly integrated.* For the elastic model, this observation was first made by Truesdell in the fifties, and it remains relevant in the inelastic case. Of course things are not so simple because the coefficients in the equations become more and more tricky as the degree of the polynomial increases, so if one wants precise estimates one has to be clever to keep these coefficients under control.

It is a classical result in probability theory that a distribution function which decays fast enough is uniquely determined by all its polynomial moments. So, as long as one remains in the class of distribution functions which decay very fast (and in particular have all their moments finite), one can in principle integrate "explicitly" the spatially homogeneous Boltzmann equation with Maxwellian kernel. Information about the asymptotic behavior can also be obtained in this way: if one proves that all polynomial moments of f converge, as time goes to infinity, to the corresponding

polynomial moments of some distribution f_∞, this means that f does converge to f_∞, at least in weak sense (in the sense of convergence of observables).

2.5. **Fourier transform.** Let the Fourier transform of f (with respect to the velocity variable) be defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(v) e^{-i\xi\cdot v}\, dv.$$

To compute the time-evolution of the Fourier transform, one is again led to evaluate

$$\int Q^+(f,f) e^{-i\xi\cdot v}\, dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} f f_* \left(\int_{S^2} e^{-i\xi\cdot v'}\, d\sigma \right) dv\, dv_*.$$

Now, from (3),

$$\int_{S^2} e^{-i\xi\cdot v'}\, d\sigma = e^{-i\xi\cdot\left(\frac{v+v_*}{2}\right)} e^{-i\left(\frac{1-e}{4}\right)\xi\cdot(v-v_*)} \int e^{-i\left(\frac{1+e}{4}\right)(\xi\cdot\sigma)|v-v_*|}\, d\sigma.$$

(The reader should not confuse the e standing for exponential, and the one standing for the restitution coefficient.) The factor in front of the integral is of the form $e^{iv\cdot k} e^{iv_*\cdot k_*}$, which is good; but the expression inside the integral is more ugly. However, in the integration process, the particular direction of ξ should not matter very much, since it enters the integral only through $\xi\cdot\sigma$, and σ varies over the sphere. As noticed by Bobylev in the seventies, a simple symmetry argument allows one to replace the expression $(\xi\cdot\sigma)|v-v_*|$ inside the σ-integral by $|\xi|\sigma\cdot(v-v_*)$. Then one can write

$$\int_{S^2} e^{-i\xi\cdot v'}\, d\sigma = \int_{S^2} e^{-iv\cdot k} e^{-iv_*\cdot k_*}\, d\sigma,$$

where k and k_* are certain frequency vectors that depend only on ξ and σ. Then integrate against $f f_*\, dv\, dv_*$, and exchange integrals, to come up with

$$\int Q^+(f,f)\, e^{-i\xi\cdot v}\, dv = \int_{S^2} \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} f f_* e^{-iv\cdot k} e^{-iv_*\cdot k_*}\, dv\, dv_* \right) d\sigma = \int_{S^2} \widehat{f}(k)\widehat{f}(k_*)\, d\sigma.$$

(For careful such computations in the elastic case, consider, e.g. [1, Appendix].)

To better appreciate what has been achieved, let us rewrite the corresponding equation for $\widehat{f}$ in the case of a constant collision kernel of unit integral (forgetting the temperature dependence for simplicity):

$$\frac{\partial \widehat{f}}{\partial t} + \widehat{f} = \frac{1}{4\pi} \int_{S^2} \widehat{f}(\xi^+)\widehat{f}(\xi^-)\, d\sigma,$$

where

$$\xi^\pm \;=\; \frac{\xi}{2} \pm \left[\left(\frac{1-e}{4} \right) \xi + \left(\frac{1+e}{4} \right) |\xi|\sigma \right].$$

This looks like a Boltzmann equation, except that the integral is only over S^2, not over $\mathbb{R}^3 \times S^2$! The $\mathbb{R}^3$ integral has been absorbed into the definition of Fourier transform. As a consequence, this equation is much simpler and reveals convenient for

- studying moments (polynomial moments are derivatives in Fourier space at frequency vector $\xi = 0$, and it is often more convenient to manipulate Taylor expansions than moment expansions);

- study the regularity of the Boltzmann equation (regularity is all about control of the amplitude of high Fourier modes);

- linearizing and computing eigenfunctions and eigenvalues;

- looking for exact equilibrium or self-similar solutions. (Think that Fourier transform is the standard method for the study of the central limit theorem, which is all about the asymptotic behavior of sequences of densities obtained by successive convolutions.)

2.6. **Some results.** In the heat bath problem, moment equations have been established by Carrillo, Cercignani and Gamba [22], then used to prove the convergence to equilibrium by Bobylev and Cercignani [14]. This is a neat application of the methods described before.

The problem of self-similar profiles is more subtle. In dimension $d = 1$, there is a simple, explicit Homogeneous Cooling State, or "HCS" (see for instance [5, Section 2.1]):

$$f(t,v) = \left(\frac{2}{\pi\sqrt{T(t)}}\right) \frac{1}{\left(1 + \dfrac{v^2}{T(t)}\right)^2}.$$

Recall that there is no equivalent for the elastic case, since the elastic collision operator is trivial in dimension 1.

In dimension 3, the situation is more complicated. Let us assume for simplicity that the restitution coefficient e is constant. The rescaled equation for HCS

$$Q(f,f) - \nabla_v \cdot (fv) = 0,$$

written in Fourier variables, always admits a unique physically relevant solution[1]. As conjectured by Ernst and Brito, this solution has a *very fat tail*, decaying like an inverse power law in the velocity variable (with a complicated exponent). This solution is often called the **Ernst-Brito solution**. Up to changing the origin of time, we may assume that it has temperature 1 at initial time. The relevant mathematical reference here is the work by Bobylev and Cercignani [14] who completely justified the conjecture of Ernst and Brito.

Once we know there is a unique HCS, the natural question is whether it is attractive. Thus we go to self-similar variables and study convergence to equilibrium. Using Fourier techniques again, Bobylev and Cercignani [14] did prove convergence in the rescaled variables, under certain assumptions on the initial datum which were later removed by Bobylev, Cercignani and Toscani [15]. The final result is as follows [15, Theorem 5.1]. Let $f(t,v)$ be a solution of the spatially homogeneous inelastic Maxwellian Boltzmann equation with constant restitution coefficient $e \in [0,1)$. Assume that the initial datum f_0 has unit temperature, zero mean momentum, and

[1]Another family of solutions was described earlier by Bobylev, Carrillo and Gamba [12]; these solutions have rapid decay, but they are not nonnegative for (at least) almost all values of e. It was mistakenly stated in [14] that there is a countable set of "exceptional" (resonant) values of e for which the two families coincide; but such a statement appeared to be in contradiction with the positivity of the Ernst-Brito solutions. Thus, the physically relevant self-similar solutions have power-like tails for *all* values of e. This issue is clarified in the last section of Bobylev and Gamba [16]. (All of this was explained to me by Bobylev.)

let F be the associated self-similar Ernst-Brito profile. Further assume that f_0 has a finite moment of order $s > 2$. Then, for a well-chosen parameter $\mu > 0$,

$$e^{-3\mu t} f(t, v e^{-\mu t}) \longrightarrow F(v),$$

in the sense of weak convergence (convergence of observables).

It is interesting to note that, in contrast with the heat bath problem, the method of moments is not applicable here, since the asymptotic state has only slow decay – yet Fourier transform continues to be useful.

Let me conclude with some variants:

- I am not aware of a decisive study for variable restitution coefficient. Bobylev, Carillo and Gamba [12, Section 6.2] have considered the case $e = e(T)$ under the condition that

$$\lim_{T \to 0} \frac{1 - e(T)}{T^\alpha} \in (0, \infty), \qquad 0 < \alpha < 1,$$

and established an asymptotic Maxwellian self-similar behavior of the form

$$f(t, v) \simeq \frac{e^{-\frac{|v|^2}{2T(t)}}}{(2\pi T(t))^{3/2}}, \qquad t \to \infty,$$

where of course $T(t)$ satisfies the Haff law.

- The linear Boltzmann equation (for inelastic Maxwellian collisions) was studied by Spiga and Toscani [42].

2.7. **Contracting distances.** Lyapunov functionals often yield precious information about asymptotic behavior, for instance if one is interested in rates of convergence. In kinetic theory, the standard Lyapunov functionals are related to entropy and Boltzmann's H Theorem; but there is no H Theorem for inelastic Boltzmann equation.

In the seventies, Tanaka discovered a new Lyapunov functional for the spatially homogeneous Boltzmann equation with Maxwellian kernel. He actually found more: a distance in which any two solutions of the Boltzmann equation become closer:

$$t' \geq t \Longrightarrow \quad d\big(f(t', \cdot), g(t', \cdot)\big) \leq d\big(f(t, \cdot), g(t, \cdot)\big).$$

Tanaka's distance is a well-known object in probability theory, also called Monge-Kantorovich or Wasserstein distance of order 2: whenever μ, ν are any two probability measures in $\mathbb{R}^d$, define

$$W_2(\mu, \nu) = \sqrt{\inf \mathbb{E}|X - Y|^2},$$

where $\mathbb{E}$ stands for expected value, and the infimum is taken over all random variables X and Y with respective laws μ and ν. This amounts to look for a *coupling* of μ and ν with maximum covariance. It turns out that W_2 satisfies the axioms of a distance (triangular inequality, etc.)

In the nineties, Toscani and collaborators found other contracting distances which are simpler to use, of the form

$$d_s(\mu, \nu) = \sup_{\xi \in \mathbb{R}^d} \frac{\big|\widehat{\mu}(\xi) - \widehat{\nu}(\xi)\big|}{|\xi|^s}.$$

This is well-defined (finite) as soon as μ, ν have the same polynomial moments up to order $\lceil s \rceil - 1$ (this means in particular that μ and ν should have same mean value

if $s = 2$; the same mean value and covariance matrix if $2 < s \leq 3$). The behavior of these distances with respect to the Boltzmann equation has been studied by Toscani together with various researchers such as Gabetta, Wennberg or the author. Note that related distances have proved to be useful for the Navier-Stokes equation, in particular after the work of Le Jan and Sznitman [36] (related issues are discussed in the recent review paper by Cannone [20]).

These techniques were recently adapted by Bisi, Carrillo and Toscani [9] who proved that d_2 is contracting along solutions of both the spatially homogeneous inelastic Boltzmann equation with Maxwell molecules, either with or without a heat bath Δ_v term. In the presence of Δ_v, this is even a *strict* contraction, in the sense that one has an equation like

$$\frac{d^+}{dt}\, d_2\big(f(t), g(t)\big) + K\, d_2\big(f(t), g(t)\big) \leq 0,$$

with d^+/dt standing for the upper right-derivative. Of course this implies that f and g become exponentially closer as $t \to \infty$; by choosing g to be the stationary state one deduces that the convergence to equilibrium is exponentially fast.

Convergence here is again in the sense of weak convergence, however it is a general rule that the combination of (i) exponential convergence in the sense of such a distance, (ii) uniform in time moment bounds, and (iii) uniform in time regularity bounds in Sobolev spaces (say $\int |\xi|^{2k} |\widehat{f}(\xi)|^2\, d\xi$) imply exponential convergence in strong sense. For instance, if moments and regularity bounds are as strong as possible, then all derivatives of f converge uniformly in v, and exponentially fast, towards the corresponding derivatives of the equilibrium. Thus the convergence is as smooth as one can hope for; moreover the rate of convergence is almost as good as in the weak distance.

In the case without heat bath, one may also wish to use this technique to study the relaxation to the Ernst-Brito HCS, and then rephrase the convergence results of Bobylev, Cercignani and Toscani in the setting of Fourier distances. This is not just for the sake of formalism: contractive distances give strong information (about nonlinear stability, for instance). But now d_2 is no longer a strict contraction; this motivates the use of a stronger distance d_s for $s > 2$. There is a recent work on this topic by Bisi, Carrillo and Toscani; although computations are quite more tricky, the result is similar, and exponential convergence to equilibrium can be proven in this way. Exponential convergence in the rescaled variables, when translated back in the original time variables, means that there is an improvement of the *power* of t^{-1} in the convergence rate when one replaces the Dirac mass by the Ernst-Brito solution:

$$d\big(f(t), \delta_0\big) = O(t^{-1}), \qquad d\big(f(t), f_{\mathrm{EB}}(t)\big) = O(t^{-(1+\alpha)}), \qquad \alpha > 0.$$

Two remarks are in order about this result. First, the exponent α which is found by Bisi, Carrillo and Toscani agrees with the one which appears in previous computations by Bobylev, Cercignani and Toscani [14, 15]; and the result of convergence to equilibrium is not new either; the main progress here is that this is recast in the form of a true contraction estimate. Secondly, to deduce a decay result "in physical space", one would need to have some a priori bounds on the solutions, which do not seem to be known at present.

Here are some further comments. Li and Toscani have applied the tools of Wasserstein distances to variants of the McNamara-Young equation [37], in dimension 1. But it would also be interesting to check whether the original Tanaka distance is also nonexpanding in the inelastic case. This might be of interest if one has in mind to do something about it in a spatially inhomogeneous context, since Tanaka's distance may be easier to use in that context (see for instance Carlen and Gangbo [21]). Further note the neat identity

$$\sqrt{T} = W_2(f, \delta_u) \qquad u = \int_{\mathbb{R}^3} f(v)\, v\, dv,$$

which suggests that W_2 is somewhat natural in that context. A result of convergence in Wasserstein sense would be judged by many authors to be physically relevant in itself, while Fourier-based metrics can be criticized in that respect.

In practical problems, W_2 might be plotted from experimental data, since it can be computed explicitly for radially symmetric functions:

$$W_2(f_1, f_2) = \sqrt{\int_0^1 \left|F_1^{-1}(t) - F_2^{-1}(t)\right|^2 dt},$$

where $F^{-1} \circ F$ is the identity, and

$$F_i(x) = (4\pi) \int_0^x f_i(r) r^2\, dr.$$

A lot of information about Wasserstein distances can be found in [45, Chapter 7]; see also Chapter 2 in the same reference.

2.8. **Information theory.** The natural tendency of collisions is to kill information: By adding some kind of randomness in the system, collisions make it more and more difficult to estimate parameters of the distribution, or to reconstruct the initial distribution. One can interpret in this way the trend to Gaussian as a tragic loss of information. For Maxwell collision kernels, this is related to certain information-theoretical inequalities: for instance, $H(Q^+(f,f)) \leq H(f)$, where $H(f) = \int f \log f$ is the familiar H-functional; a similar inequality holds true with the Fisher information (for all that see [44, Chapter 4, Section 3]).

Inelastic collisions however create correlations (information) at the same time as the destroy it. It is not so clear in which sense this statement should be taken, and maybe one can establish some variants of these information-theoretical inequalities, that would help understanding better the evolution of information in a granular gas.

2.9. **Conclusion.** To summarize, one can fairly say that during the period 2000-2005 the most important questions about the spatially homogeneous Maxwell model of granular gas have been solved, and that there is a rather good understanding of the evolution of temperature and moments, the asymptotic behavior as $t \to \infty$, and the tail behavior. Of course there is room for refinements, but the priority now for the Maxwell model is certainly to tackle inhomogeneous situations.

3. Gradient flow structure

Among the simplified models of granular media that were mentioned before is

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \left(f(v) \int f(v_*) |v - v_*| (v - v_*)\, dv_* \right) + T_e \Delta_v f + \alpha \nabla_v \cdot (fv) \qquad (T_e, \alpha \geq 0) \tag{10}$$

where the first term on the right-hand side is a phenomenological model for inelastic collisions (with a cooling effect), similar to the McNamara-Young operator; the second one describes a heat bath; and the last one is a friction term.

Complicated as it may seem, equation (10) has a particular structure: it can be rewritten

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \left(f\, \nabla_v \frac{\delta \mathcal{F}}{\delta f} \right), \tag{11}$$

where the functional $\mathcal{F}$ reads

$$\mathcal{F}(f) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(v) f(v_*) \frac{|v - v_*|^3}{3}\, dv\, dv_* + T_e \int_{\mathbb{R}^d} f \log f + \alpha \int_{\mathbb{R}^d} f \frac{|v|^2}{2}\, dv$$

and $(\delta \mathcal{F})/(\delta f)$ is defined by the identity

$$D\mathcal{F} \cdot (\delta f) = \int \frac{\delta \mathcal{F}}{\delta f}\, \delta f,$$

where $D\mathcal{F}$ is the differential of $\mathcal{F}$ and δf is an infinitesimal variation of f. In other words, an infinitesimal variation of $\mathcal{F}$ along an infinitesimal change δf of the density f can be computed by integrating $\delta \mathcal{F}/\delta f$ against δf.

Equations of the type (11) are a gradient flow in a very precise sense. What is a gradient flow? It is an equation of the form

$$\frac{dX}{dt} = -\operatorname{grad} \mathcal{F}(X(t)),$$

where X is an unknown living in a "manifold" $\mathcal{M}$ (the phase space of the system), $\mathcal{F}$ is an "energy" defined on $\mathcal{M}$, and grad is the "gradient" operator, which is associated to $\mathcal{F}$ by the relation

$$D\mathcal{F}(X) \cdot (\delta X) = \big\langle \operatorname{grad} \mathcal{F}(X), \delta X \big\rangle_X.$$

Here the left-hand side is the infinitesimal variation of $\mathcal{F}$ at X under an infinitesimal variation δX, and the right-hand side is the scalar product of grad $\mathcal{F}(X)$ and δX. Both grad $\mathcal{F}(X)$ and δX should be thought of as vectors in the tangent space to $\mathcal{M}$ at X, and the scalar product $\langle \cdot, \cdot \rangle_X$ should be specified. To summarize, a gradient flow is given by a manifold $\mathcal{M}$, an energy functional $\mathcal{F}$ on $\mathcal{M}$, and a "Riemannian structure", which is a family of scalar products defined (smoothly) on each tangent space to $\mathcal{M}$. It so happens that gradient flows occur everywhere in physics and mathematics.

In the case which is of interest to us, equation (11) can be formally rewritten as

$$\frac{\partial f}{\partial t} = -\operatorname{grad} \mathcal{F}(f),$$

where $\mathcal{M}$ is the space of probability measures, equipped with the following "Riemannian" structure: whenever δf is a small variation of density (so δf is a function

of v), its square norm, as measured by the scalar product at "point" f, is

$$\|\delta f\|^2 = \inf \left\{ \int f(v)|u(v)|^2\, dv; \quad \delta f + \nabla_v \cdot (fu) = 0 \right\}.$$

Here is how one can understand this formula: think of velocities v as being positions in phase space; particles are distributed according to density $f(v)\, dv$, and they perform certain infinitesimal movements in such a way that the density varies from a certain amount δf. You don't know the velocities $u = \dot{v}$ (velocity in phase space) of the particles, but you can try to guess: these velocities should be compatible with the observed variation δf, so the continuity equation $\delta f + \nabla_v \cdot (fu) = 0$ should be satisfied. Among all admissible vector fields, select one which is most economical, in the sense of having minimum kinetic energy.

This structure has been studied at length by many authors, starting from the pioneering work of Otto around 2000. It is intimately related to the quadratic Wasserstein distance W_2: in fact W_2 is nothing but the associated geodesic distance. A lot of information about that can be found in [45, Chapter 8].

Why are we doing all that? One advantage of identifying a gradient flow structure is that it yields interesting recipes for computing, say derivatives of functionals along the flow, in terms of gradients and Hessians; this is what is described in [45, Chapter 8] as *Otto's calculus*. Another advantage of such a formalism is that the convexity properties of the energy functional might help the study of convergence to equilibrium. For instance, if $\mathcal{F}$ is λ-uniformly convex (i.e. the Hessian operator, in an appropriate sense, is greater than $\lambda\, \mathrm{Id}$), then trajectories get closer to each other like $O(e^{-\lambda t})$, and there is exponential convergence to the unique infimum f_∞ of $\mathcal{F}$. This also comes with automatic inequalities such as

$$\frac{\lambda}{2} W_2(f, f_\infty)^2 \leq \mathcal{F}(f) - \mathcal{F}(f_\infty) \leq \frac{1}{2\lambda} \| \mathrm{grad}\ \mathcal{F}\|^2 = -\frac{1}{2\lambda} \left(\frac{d\mathcal{F}}{dt} \right).$$

One has to be careful: convexity of a functional is here understood as convexity along geodesics of the induced structure. For instance $\int f|v|^2\, dv$ should be considered as strictly convex, and $-\int f|v|^2\, dv$ as strictly concave! (although both are formally linear functionals of f...) The technical word is "displacement convex", which means more or less that the functional is convex along variations of densities which correspond to particles going in straight lines.

In the present case, the convexity of the "interaction potential" $|v - v_*|^3$ guarantees that the first term in $\mathcal{F}$ is displacement convex; it is known from the work of McCann that the second term is also displacement convex; finally the last term is also displacement convex (uniformly if $\alpha > 0$).

Using this apparatus and some work, Carrillo, McCann and the author [24, 23] studied in some detail the convergence to equilibrium for (10). The convergence itself was already established by Benedetto, Caglioti, Carrillo and Pulvirenti [6] (in dimension 1 only), so the novelty lied in the derivation of explicit bounds on the rate of convergence.

The conclusions can be summarized roughly as follows: For $T_e > 0$ (heat bath case) there is always a unique equilibrium (up to translation if $\alpha = 0$). If $\alpha > 0$ then the convergence is at least like $O(e^{-\alpha t})$. If $\alpha = 0$ there is still exponential convergence, and one can obtain a lower bound on the rate of convergence. The main problem here is to overcome that lack of convexity of the cubic potential close

to $v = v_*$; the key thing is that the heat bath forces the velocity distribution to be spread out enough that the system is not concentrated too much (if two particles have very close velocities v and v_*, so that they interact quite weakly, they can still interact with particles of velocity w which is far both from v and v_*; this mechanism of indirect interaction leads to an efficient relaxation even though the convexity is degenerate).

All these results are naturally expressed in terms of W_2 distance, but they can be combined with regularity estimates to obtain convergence in, say, supremum norm, under just an assumption of finite kinetic energy (it is interesting to note that $\mathcal{F}(f)$ will become finite for any positive time even if it is infinite at time 0; this can be interpreted as a parabolic regularization effect, but is best thought of as a manifestation of the displacement convexity of the functional). For more details the reader can refer to [24, 23].

Let me note in passing that the existence of these models of granular media is a true gift to us mathematicians. Indeed, this is one of the very rare examples available in which one encounters an energy functional $\mathcal{F}$ containing an interaction potential with a *convex* interaction potential (of course this is possible here because the phase space is really a velocity phase space, and interaction *increases* with the difference of velocities, while it should *decrease* with the difference of positions). Without this example, part of the theory of gradient flows with interaction potential would have had (so far) essentially no case of application.

Remark. In the case without heat bath or friction, one may wish to study the relaxation to HCS by going to self-similar variables; then the equation is

$$\frac{\partial f}{\partial t} = \nabla_v \cdot (f\, \nabla_v (f * |v|^3/3)) - \nabla_v \cdot (fv),$$

and one is in deep trouble since the antidrift is displacement *concave*. The equilibrium for the rescaled equation is very simple: a combination of two Dirac masses, $f_\infty = (\delta_{-1/2} + \delta_{1/2})/2$. A linear stability analysis cannot really say anything because the tangent space is extremely degenerate. Actually, we shall see later that in general, the convergence is *not* exponential.

4. One-dimensional rigidity

Reduction of dimension leads to simplified models with more estimates, and less room to behave strangely. This might be a good or a bad thing. In this section I shall discuss briefly two examples in which a one-dimension assumption leads to additional estimates; and in both cases the conclusion might be that the model is to a certain extent inappropriate.

4.1. **Clustering in dimension 1 is difficult.**

Benedetto and Pulvirenti [8] studied the possibility of clustering in dimension 1 of velocity and dimension 1 of space:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = Q(f, f),$$

where $Q(f, f)$ is an inelastic hard sphere collision operator, with a restitution coefficient given by

$$e(|v - v_*|) = \frac{1}{1 + a|v - v_*|^\beta}, \qquad 0 < \beta < 1. \tag{12}$$

The possibility of clustering is related to the amount to which the H Theorem is violated, so one may study the entropy dissipation (as opposed to entropy production) and try to bound above $\dot{H}$ above, where $H(t) = \int f(t,\cdot) \log f(t,\cdot)$. It is easy to show that

$$\dot{H}(t) \leq \frac{1}{2} \int dx\, dv\, dv_*(J-1)|v - v_*| f f_*. \tag{13}$$

The right-hand side looks a bit like $\int \rho^2 \sqrt{T}$, where ρ and T respectively stand for density and temperature, so there is no a priori control on this right-hand side from just basic conservation laws.

At this point Benedetto and Pulvirenti use a variant of a trick introduced by Bony and adapted by Cercignani: they consider functionals of the form

$$I_\alpha(t) = \int_{x<y} dx\, dy \int dv\, dv_*\, \varphi_\alpha(v - v_*)\, f(t,x,v)\, f(t,y,v_*),$$

where $\varphi_\alpha(z) = \text{sign}\,(z)|z|^\alpha$. In contrast with the right-hand side of (13), the functional I_α is essentially well-controlled: if one forgets about φ_α it looks like $(\int \rho)^2$, which is well-controlled thanks to just basic conservation laws. In that respect the condition $x < y$ in the integration does not change anything; but it will have an important impact when one computes the *time-derivative* of I_α under solutions of the Boltzmann equation. Indeed, the action of the streaming term $v\partial_x f$ will result, after integration by parts, in the appearance of a Dirac function $\delta_{y=x}$, and then we obtain something that looks like the right-hand side of (13). Here is a more precise formulation: define

$$\mathcal{I}_\alpha(T) = \int_0^T \left(\int dx\, dv\, dv_* |v - v_*|^{1+\alpha} f f_* \right) dt,$$

then there are some relations between I_α and $\mathcal{I}_\alpha$:

$$\mathcal{I}_\alpha(T) \leq I_\alpha(0) - I_\alpha(t) + C\mathcal{I}_{(1+\beta)\alpha}(T),$$

where C is a constant depending on α, whose value is 0 if $(1+\beta)\alpha = 1$.

When comparing the expressions for I_α and $\mathcal{I}_\alpha$, one sees that in effect what has been done is the replacement of a t-integral by an x-integral; obviously this is a particular feature of dimension 1.

Benedetto and Pulvirenti use this trick to get a control of $\mathcal{I}_\alpha$, and finally of $\dot{H}$. Then they easily conclude that, at least for small total mass, there is no blow-up (clustering) for the model. The reader may consult [8] for much more information.

Since this conclusion is at variance with results of numerical simulations for granular media, one can consider the result as physically irrelevant, and wonder where the problem lies. Is it the elasticity law? For the proof to work it is sufficient that e behaves like (12) for *small* values of $|v - v_*|$, and β really has to be strictly positive. Such constitutive laws do not a priori seem unrealistic, but maybe there is an unexpected problem here. The problem might also lie in the assumption of smallness of mass, although this would not really be expected. The final possibility is that the problem is with the one-dimensionality assumption which does not leave enough room for clustering.

4.2. **HCS are a bad approximation for the McNamara-Young model.** Now I shall consider what may be the most simple phenomenological kinetic model for granular medium: the spatially homogeneous McNamara-Young equation in dimension 1,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(\frac{\partial}{\partial v}\left(f * \frac{|v|^3}{3}\right)\right).$$

To fix ideas, assume that the mean velocity is 0. As time goes to infinity, $f(t,\cdot)$ converges to δ_0 like $O(t^{-1})$. The question now is whether $f(t,\cdot)$ is better approximated by the HCS, which in that case is just

$$S(t) = \frac{1}{2}\left(\delta_{-\frac{1}{2t}} + \delta_{\frac{1}{2t}}\right).$$

To study that problem, let us go to rescaled variables, which adds an anti-drift term:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}\left(f\,\frac{\partial}{\partial v}\left(f * \frac{|v|^3}{3}\right)\right) - \frac{\partial}{\partial v}(fv), \tag{14}$$

and then the HCS transforms into the stationary solution $S = (\delta_{-1/2} + \delta_{1/2})/2$.

It was proven by Benedetto, Caglioti and Pulvirenti [7] that there is indeed convergence, in rescaled variables, to the stationary state. This conveys the idea that the HCS is indeed a better approximation to the solution. However, it was later proven by Caglioti and the author [19] that the convergence is actually very poor: essentially, it can be at best like $O(t^{-1})$ in the rescaled variables, which means a *logarithmic* in time improvement at the level of the original variables. This is actually a good example in which a quantitative result essentially kills a qualitative one.

Here is an idea of the argument. Equation (14) has the form of a continuity equation, so it can be formally solved by the characteristic methods (in velocity space): $f(t)$ can be interpreted as the velocity distribution for a bunch of particles whose velocity evolves according to

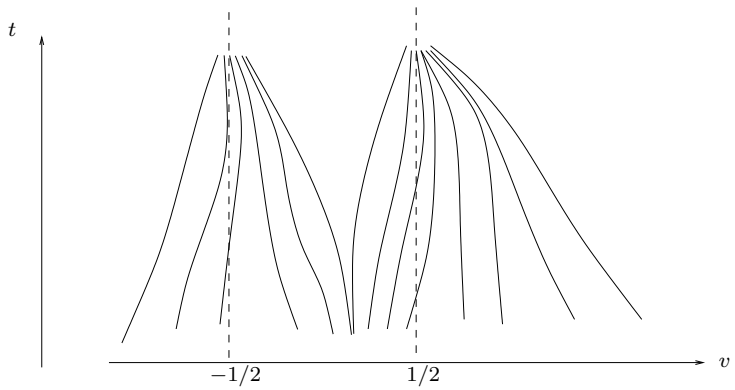
$$\frac{dv}{dt} = \xi(v) \equiv v - \int_{\mathbb{R}} (v - v_*)|v - v_*| f(t, v_*)\, dv_*.$$

Since we are in dimension 1, the associated flow is *order-preserving*: particles with a higher velocity always keep a higher velocity than particles with a lower velocity. The vector field ξ is a function on $\mathbb{R}$ which goes to $+\infty$ for $v \to -\infty$, $-\infty$ for $v \to +\infty$, and is successively decreasing, increasing, decreasing again. Its derivative is maximum at the median of f. Since ξ is an integral expression of f and vanishes for $f = S$, it is plausible that ξ can be controlled in terms of some weak distance of f to S; actually one can show that

$$\frac{\partial \xi(t,v)}{\partial v} \leq 2\, W_2\big(f(t,\cdot), S\big).$$

Now what happens as $t \to \infty$? Since the flow is order-preserving, the left-half of the particles should go to velocity $-1/2$, while the right-half should go to velocity $+1/2$ (Possible trajectories in (t,v) variables are represented on the figure below).

If there are some particles that are around the median of f, it will be necessary to *separate* them, resulting in a considerable stretching in velocity space. But the maximum amount of stretching is controlled by an upper bound on the divergence of the vector field, which in the present case means an upper bound on the derivative of ξ, and we have seen that this derivative is controlled by the distance of f to S. So when the solution approaches S, the amount of stretching which is available



vanishes, precisely when a strong stretching should take place! Combining all these elements, it is possible to show that

$$\int_0^T W_2\big(f(t,\cdot),S\big)\,dt \geq K\log T \tag{15}$$

as $T \to \infty$.

A similar estimate holds true even if there is no mass close to the median of f: in that case one can use the property that a very strong negative divergence of the flow is necessary to concentrate mass around the points $-1/2$ and $+1/2$. Consult [19] for more details.

The bound (15) prevents any convergence rate like $O(t^{-(1+\epsilon)})$, no matter how small ϵ is. In this sense the convergence has to be very slow, and, as I said above, in the original variables $W_2(f(t), S_t)$ cannot be much smaller than $O(1/(t\log t))$, while $W_2(f(t),\delta_0)$ is $O(1/t)$. The physical relevance of this conclusion seems to be corroborated by numerical simulations of A. Barrat, indicating that the convergence to self-similar solutions is indeed poor for large times.

Once again it is not clear where the problem lies. Maybe this is due to the oversimplified form of the HCS, or the fact that it has a singular part, and some vacuum regions (not all velocities lie in the support). Maybe this is also due to the one-dimensional nature of the equation.

5. True inelastic hard spheres

To caricature, mathematical research on the inelastic Boltzmann equation focused on baby models from 1997 to 2000, then turned to the Maxwell model, and around 2004 appeared the first works devoted to the true inelastic hard spheres model, while staying spatially homogeneous. Here below is a rather exhaustive list of references:

- Gamba, Panferov and the author [32] set up the framework and devised several techniques that would be used again in the sequel;

- Bobylev, Gamba and Panferov [17] proved an important technical result for the study of tail behavior in integrated version;

- Mischler, Mouhot and Rodriguez Ricard [39, 40] sharpened and generalized all the previous results; in these works are considered for the first time general restitution constitutive laws with a dependence of e upon T or $|v - v_*|$;

- In a still unpublished sequel to their first work, Gamba, Panferov and the author pushed some maximum principle techniques to study pointwise tail estimates.

- Some results about the linear case were obtained by Lods and Toscani [38] (they guessed the form of the equilibrium by first replacing the model by its grazing collision limit).

In this section, I shall informally describe some of the techniques and results that can be found in these works. For simplicity, I shall assume the restitution coefficient to be constant, but in most cases this assumption can be relaxed.

5.1. **Energy dissipation.** Consider the spatially homogeneous inelastic Boltzmann equation; fix the mean momentum to be zero, and as usual assume that the total mass is normalized to 1. It is easy to show that the energy dissipation due to collisions is

$$K \int f f_* |v - v_*|^3 \, dv \, dv_* \qquad (K > 0).$$

Thus the energy does not satisfy a closed equation as in case of Maxwell collisions. Yet something can be done: by *Jensen's inequality*, applied with the convex function $|\cdot|^3$ and the probability measure $f(v)\,dv$,

$$\int f(v_*)|v - v_*|^3 \, dv_* \geq \left| v - \int f_* \, v_* \, dv_* \right|^3 = |v|^3.$$

Then, by *Hölder's inequality* (or Jensen's inequality with the convex function $x^{3/2}$),

$$\int f(v)|v|^3 \, dv \geq \left(\int f(v)|v|^2 \, dv \right)^{3/2}.$$

Then we can write the temperature equation in the form

$$\frac{dT}{dt} \leq -KT^{3/2} + \text{contributions from noncollisional terms.}$$

This results in two main consequences: First, a priori estimates on the temperature: the temperature is always controlled uniformly in time, be it for the "raw" Boltzmann equation or in presence of a heat bath, an anti-drift or a shear flow additional term. Secondly, for the plain Boltzmann equation we recover a half of Haff's law (if I may say), namely

$$T(t) \leq \frac{C}{t^2}.$$

The other half is true also, as will be seen later.

5.2. **Moment estimates.** Moment equations can still be written, but now they are not closed any longer. Yet inequalities can be hoped for. If we are interested in the moment of order $s > 2$, then the relevant quantity is

$$(|v'|^s + |v'_*|^s) - (|v|^s + |v_*|^s).$$

Povzner inequalities have been used for many years in kinetic theory to bound this from above; see [44, Chapter 2, Section 2.2]. The search of precise tail estimates motivated the development of refined such inequalities, taking into advantage the integration over the collision parameter σ. The following estimate, due to Bobylev in

the elastic case, was obtained by Bobylev, Gamba and Panferov [17] in the inelastic setting: for any integer $p \geq 1$,

$$\frac{1}{4\pi} \int_{S^2} (|v'|^{2p} + |v'_*|^{2p} - |v|^{2p} - |v_*|^{2p})\, d\sigma \leq -(|v|^{2p} + |v_*|^{2p}) + \gamma_p (|v|^2 + |v_*|^2)^p, \qquad \gamma_p = O\left(\frac{1}{p}\right). \tag{16}$$

Although it can be formulated in very elementary terms and does not involve any partial differential equations, this estimate is *very* tricky. A nice observation made in [17] to simplify the proof is the following: whenever φ_1 and φ_2 are nondecreasing functions on $[-1, 1]$, and α_1, α_2 are given unit vectors, then

$$\int_{S^2} \varphi_1(\alpha_1 \cdot \omega)\, \varphi_2(\alpha_2 \cdot \omega)\, d\omega$$

is maximum when $\alpha_1 = \alpha_2$. The rest of the proof, unfortunately, is plenty of heavy computations and convexity arguments.

Why is (16) interesting? Expand the last term using the binomial formula, then the terms of higher degree, $|v|^{2p}$ and $|v_*|^{2p}$, appear with a positive coefficient γ_p, which is negligible in front of the negative coefficient -1 at the beginning of the right-hand side of (16). All the terms that remain have the form $O(|v|^{2\ell}|v_*|^{2p-2\ell})$ for some positive integers ℓ. Fix such a term and integrate the whole (16) against $f(v)f(v_*)|v - v_*|\, dv\, dv_*$ to compute the rate of change of the moment of order $2p$: the contribution of that term is bounded above by something like

$$\int f(v)f(v_*)(|v| + |v_*|)|v|^{2\ell}|v_*|^{2p-2\ell}\, dv\, dv_*,$$

which in turn can be bounded in terms of moments of order 2ℓ, $2\ell + 1$, $2p - 2\ell$ and $2p - 2\ell + 1$, all of which are *strictly* less than $2p$. It is not so for the contribution of the first term in the right-hand side of (16), which leads to a negative term involving moments of order $2p + 1$. To summarize, (16) implies differential inequalities of the form

$$\frac{dM_{2p}}{dt} \leq -K M_{2p+1} + C \sum_\ell \big(M_{2\ell} M_{2p+2\ell-1} + M_{2\ell+1} M_{2p+2}\big),$$

where M_s stands for the moment of order s. From Jensen's inequality again, this implies

$$\frac{dM_{2p}}{dt} \leq -K M_{2p}^{1+(2p)^{-1}} + C \sum_\ell \big(M_{2\ell} M_{2p+2\ell-1} + M_{2\ell+1} M_{2p+2}\big),$$

and after a bit of work one actually deduces

$$\frac{dM_{2p}}{dt} \leq -K M_{2p}^{1+(2p)^{-1}} + C,$$

where the constant C and K may depend on the kinetic energy, but not on higher moments.

This has two consequences. First, for each fixed $s > 2$, the moment of order s becomes immediately finite even if it is infinite at time 0 (so there is instantaneous destruction of fat tails; in the elastic case this observation was first made by Desvillettes in the beginning nineties). Secondly, by using the precise form of (16), one can

refine the computations to get bounds on (stretched) exponential moments, since

$$\int f(v) e^{a|v|^s}\, dv = \sum_{k=0}^{\infty} \frac{a^k}{k!} \left(\int f(v) |v|^{sk}\, dv \right).$$

(This was actually Bobylev's initial motivation, in the elastic case.)

Using this apparatus, Bobylev, Gamba and Panferov [17] prove that steady state(s) $f_\infty(v)$ satisfy

$$\exists a, A > 0; \qquad \int e^{A|v|^s} f(v)\, dv = +\infty, \qquad \int e^{a|v|^s} f(v)\, dv < +\infty,$$

where

$$s = \begin{cases} 3/2 & \text{for the heat bath case;} \\ 2 & \text{for heat bath + friction;} \\ 1 & \text{for the HCS.} \end{cases}$$

These bounds establish the validity of the conjectures about the large-velocity behavior of the stationary states, in an integrated sense. Estimates such as $\int e^{a|v|^s} f_\infty < +\infty$ may be considered as an integrated upper bound on f_∞, while estimates such as $\int e^{A|v|^s} f_\infty = +\infty$ may be considered as an integrated lower bound. The same authors also show that for the shear flow problem, steady states satisfy $\int e^{a|v|} f_\infty(v)\, dv < +\infty$ for some $a > 0$, but are unable to get a lower bound.

These estimates were proven only for the stationary solution; there is reason to believe that

- the integrated lower bound is always true;

- the integrated upper bound is propagated in time, that is, if it holds true for the initial datum then it holds true for all times;

- a weaker version of the integrated upper bound (with other exponents) automatically appears for positive times, whatever the initial datum.

5.3. **Regularity estimates.** One of the important discoveries by Lions in the nineties was the *regularity of the gain operator*: the operator Q^+ is regularizing, meaning that $Q^+(f, f)$ is typically smoother than f itself. In contrast, $Q^-(f, f)$ typically has the same regularity as f. Many versions exist under different sets of assumptions; if the collision kernel was perfectly smooth and compactly supported, avoiding frontal and grazing collisions as well with large or small relative velocity collisions, then there would be a gain of $(d-1)/2$ derivatives (1 derivative in dimension 3). For realistic operators such as the hard sphere kernel, one cannot in general prove better than just a fractional gain. See [44, Chapter 2, Section 3.4] for references and discussion. It is important to note that properties of improved smoothness can be transformed into properties of improved integrability.

This feature was recently used by Mouhot and the author [41] as the basis for the regularity theory of the spatially homogeneous Boltzmann equation with hard spheres. It is easy to understand why the property of Q^+ is useful for regularity estimates: think that Q^- resembles a multiplication operator, then solutions of the spatially homogeneous Boltzmann equation should have a regularity comparable to solutions of

$$\frac{\partial f}{\partial t} + Kf = S,$$

where S is a source term, smoother than f.

Here are the consequences that can be drawn for that: First, a priori integrability bounds which prevent blow up of the solution, as well as extinction in finite time. Next, there is a lower bound on the temperature in rescaled variables; Mischler and Mouhot [40] use that to prove the lower bound in the Haff law: for the "raw" Boltzmann equation,

$$T(t) \geq \frac{c}{(1+t)^2}.$$

In the case of the Boltzmann equation with an additional anti-drift, there is propagation of regularity, and moreover there is exponential decay of singularities: the solution $f(t)$ can be decomposed into $S(t) + R(t)$, where $S(t)$ is very smooth (of arbitrary smoothness, fixed in advance), and the remainder $R(t)$ decays exponentially fast. This is proven by Mischler and Mouhot [40]. One of the technical tools used there is an inelastic variant of the *Carleman representation* (see [44, Chapter 2, Section 4.6] in the elastic case[2]) which is a way to write the Boltzmann collision operator parameterized by the precollisional velocities $'v$ and $'v_*$.

In the case of the heat bath, there is in addition instant regularization [32].

5.4. **Maximum principle.** Moment methods and Povzner inequalities lead to *integrated* tail estimates, but what about *pointwise* estimates? Is it true that in the case of the heat bath one has

$$ce^{-A|v|^{3/2}} \leq f_\infty(v) \leq Ce^{-A|v|^{3/2}}?$$

(Ideally one could hope that such an estimate holds true with $a = A$, but for the moment this looks out of reach.)

Pointwise estimates are often associated with maximum principles. The standard maximum principle for the Laplace operator is well-known: let f be a solution of $\Delta_v f \geq 0$ in a domain Ω, with $f \leq M$ on the boundary $\partial\Omega$. Then it follows that $f \leq M$ in the whole of Ω. In other words,

$$\Delta_v f \geq 0 \implies \sup_\Omega f = \max_{\partial\Omega} f.$$

There is also a time-dependent version of this maximum principle: if $f \leq M$ at time $t = 0$, $\partial_t f - \Delta_v f \leq 0$ in Ω and $f \leq M$ on $\partial\Omega \times [0, T)$, then again $f \leq M$ in the whole of Ω and for all times $t \in [0, T)$.

Gamba, Panferov and the author [32] use this principle to show a lower bound, for the heat bath problem, of the form

$$f_\infty(v) \geq ce^{-A|v|^{3/2}}.$$

The idea is in two steps: first, the smoothness and localization implies that f is bounded below on a certain small ball $B_r(v_0)$, with center v_0 and radius r (v_0 is not known with precision, but it is localized inside a larger ball). Then one can use a comparison function $\varphi(t, v)$ which is singular at v_0, but very smooth outside $B_r(v_0)$, identically zero for $t = 0$, has the correct velocity decay as $e^{-A|v|^{3/2}}$, and such that

$$\partial_t \varphi \leq \Delta_v \varphi - K|v|\varphi.$$

In fact,

$$\varphi(t, v) = K \exp\big(-bt - a(1 + |v|^2)^{3/4}\big)$$

[2]Beware: In the published version of [44] there was a typo in the Carleman formula: the collision kernel should be $\tilde{B}$ and not B; both symbols stand for different expressions.

does the job for well-chosen parameters a, b and K.

Since on the other hand (by the trivial bound $Q^+ \geq 0$ and the easy bound $Q^- \geq K|v|f$)

$$\partial_t f \geq \Delta_v f - K|v|f,$$

the maximum principle, applied in the complement of the ball $B_r(v_0)$, implies that f stays above φ for all times. Now we have a control of f inside $B_r(v_0)$, and a control of f outside $B_r(v_0)$, and the desired bound follows. See [32] for more explanations.

When the operator Δ_v is replaced by the anti-drift, Mischler and Mouhot [40] have a variant of this argument which implies a lower bound like $Ke^{-A|v|}$ for radially symmetric stationary states (so a similar bound holds true for radially symmetric HCS in the original variables). It is likely that this argument adapts to time-dependent solutions, not necessarily radially symmetric, but the job remains to be done.

Upper bounds are more tricky because the Q^+ operator needs to be controlled by above. Here again, Mischler and Mouhot obtain good bounds by above like $O(e^{-a|v|})$ for stationary radially symmetric solutions with the anti-drift, but then the method seems not to adapt to the time-dependent case.

To go further in this direction, Gamba, Panferov and the author built on an idea suggested in [44, Chapter 2, Section 6.2], namely to use a *maximum principle for the Boltzmann operator*. Here is an informal version of this principle, stated for instance in the heat bath case. Let $f(t,v)$ satisfy

$$\frac{\partial f}{\partial t} = Q(f,f) + \Delta_v f$$

and let $\varphi(t,v)$ be a "reasonably smooth" comparison function satisfying

$$\begin{cases} f(0,v) \leq \varphi(0,v) & \text{for all } v, \\ \partial_t \varphi - Q(f,\varphi) - \Delta_v \varphi \geq 0 & \text{in } [0,T) \times \Omega, \\ f(t,v) \leq \varphi(t,v) & \text{outside } \Omega, \text{ for all } t \in [0,T). \end{cases}$$

Then $f \leq \varphi$ everywhere in $[0,T) \times \mathbb{R}^3$. Here $Q(f,\varphi)$ is defined in weak form by

$$\int Q(f,\varphi)\,\psi = \int |v - v_*| f_* \varphi (\psi' - \psi).$$

This principle looks rather similar to the usual maximum principle for the heat equation; there is however an important difference: a control of f by φ is assumed *everywhere outside* Ω, not just at the boundary $\partial\Omega$. There is a good reason for that, namely the fact that the Boltzmann collision operator is nonlocal.

In practise, to use this principle it is desirable to have nice pointwise upper bounds on the gain term $Q^+(f,\varphi)$. Such bounds can be deduced from *integrated* bounds: indeed, the spreading properties of Q^+ make it possible to recover pointwise bounds on Q^+ from integrated bounds on f. A nicer example, worked out by Gamba, Panferov and the author in the case of elastic collisions, is the neat bound

$$Q^+(f, e^{-\beta|v|^2}) \leq C \left(\int f(v) e^{\beta|v|^2}\, dv \right) e^{-\beta|v|^2}$$

(for the elastic collision operator).

In the elastic case, this approach was completely worked out and it was proven by the same authors [31] that if the initial datum is bounded above by a Maxwellian, then this property holds true for all times (this was one of the last irritating open

problems in elastic spatially homogeneous kinetic theory). A similar study is on its way in the inelastic context; although it is not complete due to technical complications, it looks safe to bet that the same method does work in the end, and shows that if the initial datum is bounded above by $Ce^{-a|v|^{3/2}}$, then $f(t,v) \leq C' e^{-a'|v|^{3/2}}$ for all times $t \geq 0$.

5.5. **Summary.** Let me now summarize the main results known to this date for the spatially homogeneous inelastic hard spheres model. I shall focus on two cases: the "heat bath", or "driven" case,

$$\frac{\partial f}{\partial t} = Q(f,f) + \Delta_v f,$$

and the free or "undriven" case, distinguishing between the unrescaled version

$$\frac{\partial f}{\partial t} = Q(f,f)$$

and the rescaled version

$$\frac{\partial f}{\partial t} = Q(f,f) - \nabla_v \cdot (fv).$$

5.5.1. *Temperature.* It is proven that

$$\begin{cases} c \leq T(t) \leq C & \text{(driven)} \\ \dfrac{c}{t^2} \leq T(t) \leq \dfrac{C}{t^2} & \text{(undriven, unrescaled)} \end{cases}$$

where c, C are positive constants depending on the initial datum.

5.5.2. *Integrated tail behavior.* It is proven that equilibrium distributions satisfy

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} f_\infty(v) e^{a|v|^{3/2}}\, dv < +\infty, \quad \int_{\mathbb{R}^3} f_\infty(v) e^{A|v|^{3/2}}\, dv = +\infty & \text{(driven)} \\ \displaystyle\int_{\mathbb{R}^3} f_\infty(v) e^{a|v|}\, dv < +\infty, \quad \int_{\mathbb{R}^3} f_\infty(v) e^{A|v|}\, dv = +\infty & \text{(undriven, rescaled).} \end{cases}$$

The lower bound should also be true for time-dependent solutions.

5.5.3. *Pointwise tail behavior.* It is very likely that we can prove

$$\begin{cases} c e^{-A|v|^{3/2}} \leq f_\infty(v) \leq C e^{-a|v|^{3/2}} & \text{(driven)} \\ c e^{-A|v|} \leq f_\infty(v) \leq C e^{-a|v|} & \text{(undriven, rescaled)} \end{cases}$$

At present, only the lower bound in the driven case was proven, while both bounds were established in the undriven case, but only for radially symmetric solutions. Moreover, the lower bound also holds true for time-dependent solutions in the driven case, and this should also be the case in the undriven case.

5.5.4. *Stationary states.* In the driven case, it is known that there exists at least one smooth (with bounded derivatives up to any order) positive radially symmetric stationary state, with fast decay at infinity. It is not known whether this state is unique. In the undriven (unrescaled) case, it is known that there exists at least one smooth positive radially symmetric self-similar HCS. It is not known whether it is unique (say, unique for a given value of the initial kinetic energy).

5.5.5. *Convergence to equilibrium.* In the driven case, it is not known whether there is convergence to equilibrium. A theorem of uniqueness of the stationary state would help, but would still not be sufficient to conclude because of the absence of known Lyapunov functional. In the undriven case, it is known that there is convergence to a Dirac mass, but it is not known whether the HCS are attractive.

5.5.6. *Conclusion.* From the qualitative point of view we have a reasonable deal of information, and predictions of physicists have been confirmed at theoretical level, although by means of a very sophisticated machinery (the original papers quoted in this text might appear quite hard to read for non-experts of mathematical kinetic theory).

Unlike for the Maxwell model, we are unable to prove uniqueness of equilibrium, and we are in bad need of a new idea to resolve the problem of the asymptotic behavior in large time. Numerical simulations indicate that the convergence should be excellent, and that the equilibrium should be unique. Apart from the possibility of finding a miraculous Lyapunov functional (maybe in the form of a variant of the contracting distances known in the Maxwell case), there is no idea in the air.

Mischler and Mouhot recently tried to tackle the problem of uniqueness of the HCS by considering a family of stationary solutions (with the anti-drift) depending on a restitution coefficient e going continuously from 1 to 0 (the coefficient in front of the anti-drift is rescaled with e in such a way as to go to 0 when $e \to 1$, so that there is a nontrivial Maxwellian equilibrium for $e = 1$). There is an H Theorem with an $O(1 - e)$ error term, which may imply the uniqueness close to $e = 1$ by a perturbative argument. Then they try to use a continuity argument letting e decrease continuously to 0, and showing that no bifurcation is possible in the family of equilibria. At a bifurcation point, there would be appearance of a new conservation law for the linearized Boltzmann equation. Such an event cannot really be ruled out a priori (after all we have already seen in the Maxwell case that some values of the restitution coefficient lead to "resonance" effects and change of qualitative properties of the HCS) but might be dismissed by a theorem of classification of conservation laws for the linearized equation, proving that only mass and momentum are preserved by collisions. Would that attempt work out in the end, it would be proven that HCS are unique, leaving some hope for a proof of convergence.

6. The future of inelastic kinetic theory?

It seems clear that the next step in the mathematical kinetic theory of granular media is the introduction of the space variable and the study of clustering or breaking of space homogeneity. In fact, for engineers there is essentially no interest in spatially independent models, and it has been a long time since the spatial structure of granular flows has been studied (The reader can consult the recent textbook by Brilliantov and Pöschel [18] for references and pointers to the physical literature.) However, even for engineers, subtle phenomena such as clustering seem to retain some mystery.

Physicists say that the HCS is very unstable, unless the gas is driven by strong boundary conditions. In a personal communication, J. Brey has suggested that there should be an inhomogeneous stable self-similar solution which would describe the local behavior of the solution (this would be a fascinating eventuality). In any case the behavior which is observed in experiments or simulations is often well-described

by a separation between a cold, high-density phase and a warm, low-density one, with pattern formation and "inverse Maxwell demon".

As for the mathematical justification of these observations, we are nowhere. The only available inhomogeneous results are the works by Benedetto and Pulvirenti [8] indicating that for certain restitution coefficient laws there is no blow-up in dimension 1. We do not know whether the collapse should be expected in finite or infinite time (in which case finite-time collapse in numerical simulations should be attributed either to measurement errors or to a breakdown of the Boltzmann model).

It seems reasonable to start the study in the case of Maxwell collisions, where convergence to HCS is proven and there are contracting distances; it also seems a good idea to start with the close-to-equilibrium regime. Discussions with C. Mouhot let me suggest the following. The linearized Boltzmann collision operator should have a nice spectral theory (even that is not so clear because of the slow decay of the HCS) when associated with either the heat bath term or the anti-drift. However, one can think that these two cases behave radically differently when one adds the transport term $-v \cdot \nabla_x$: there should be *destabilization* by the anti-drift, not by the diffusion. Would that guess be true, this would be the first hope of displaying some instability of the homogeneous description. The dependence of that effect upon the dimension should be checked.

On the contrary, in the case of the heat bath, the good spectral properties of the linearized operator should make it possible to prove stability of weak inhomogeneity: indeed, if f_0 is an initial datum that lies at distance $O(\varepsilon)$ from a spatially homogeneous datum f_0^h, then one can hope that $f(t)$ lies at distance $O(\varepsilon e^{Ct})$ from the spatially homogeneous solution $f^h(t)$; then for $t \geq T$, $f^h(t)$ should be be sufficiently close to the HCS, that linearization is possible, and then for ε small enough, also $f(t)$ would be very close to the HCS. From that point on the good linear properties can be exploited.

Getting out of the perturbative regime looks completely hopeless for the moment. After all, even in the elastic case, there is still no good nonperturbative theory of the spatially inhomogeneous Boltzmann equation, and inelasticity makes things much worse. Even the rather soft DiPerna-Lions theory of weak solutions does not extend to the inelastic Boltzmann equation! (In this direction are available only some works about the inelastic Enskog model [30].)

Some day one will have to discuss boundary conditions more precisely than is done now. While the heat bath model might be in first approximation a honest model for shaking, it seems clear that it does not describe accurately the complicated space-dependent phenomena which occur in such a process. This looks extremely delicate.

Some other day one will have to rediscuss the relevance of the kinetic description: Is there molecular chaos or not? What qualitative differences between the Boltzmann and Enskog descriptions? How to choose the correlation function in the Enskog description? Does the kinetic description break down in presence of clustering? How to couple it with hydrodynamic equations? For the most part, these questions are open from the physical point of view, and seem to arise incredible difficulties.

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